REMARKS on Chernoff’s inequality:

1. Lower tails are similar: \[ P \{ S_N \leq t \} \leq e^{-t} \left( \frac{e^t}{t} \right)^t \quad \forall 0 < t \leq \mu \] (Uniform problem)

2. Chernoff is optimal: if \( S_N \sim \text{Binom}(N, \mu) \) then \[ P \{ S_N \geq t \} \geq (\mu/t)^t \quad \forall 1 \leq t \leq N \]

3. Large deviations: when \( t \) is large, the “Poisson tail” \( \sim t^{-t} = e^{-t \log t} \) is heavier than Gaussian \( e^{-t^2/2} \)

4. Small deviations: when \( t \approx \mu \), say \( t = (1+\delta)\mu \),
   \[ e^{-\mu} \left( \frac{2\mu}{t} \right)^t = e^{-\mu} \left( \frac{e^{(1+\delta)\mu}}{1+\delta} \right)^{(1+\delta)\mu} \]
   \[ = e^{\mu(\delta - (1+\delta) \log(1+\delta))} \leq e^{-\delta^2 \mu/6} \]

   Combine upper & lower tails

   \[ \text{Coc (Chernoff’s ineq: small deviations)} \]

   \[ P \{ |S_N - \mu| \geq \delta \mu \} \leq \exp(-\delta^2 \mu/6) \quad \forall \delta \in [0, 1] \]
**APPLICATION: RANDOM GRAPHS**

**Def** Erdös-Rényi model \( G(N,p) \):  
Fix a set of \( N \) vertices.  
Connect each pair of vertices with an edge independently with prob. \( p \).  

\[
G(N,p) \text{ for } N=200, \ p=\frac{1}{40}
\]

\[
G(N,p) \text{ for } p=\frac{1}{100}
\]

- **Def** The degree of a vertex \( i \) is \( \deg(i) = \# \text{edges connected to } i \)

\[
\deg(i) = S_{N-1} \sim \text{Binom}(N-1, p) \Rightarrow \mathbb{E}\deg(i) = (N-1)p =: d
\]

"Expected degree"

- **Phase transitions**  
  \[ d = 1 : \text{giant component} \]  
  \[ d = \log n : \text{connectivity} \]  
  \( \{ \) (see Wikipedia) \( \}

**Def** a graph is \( d \)-regular if \( \deg(i) = d \ \text{\forall} \ i \)

We will show: \( d \sim \log n \) is a phase transition for regularity of \( G(n,p) \):  
\[
\begin{cases}  
  d \gg \log n \Rightarrow \text{almost } d\text{-regular} \\
  d \ll \log n \Rightarrow \text{very far from it}
\end{cases}
\]
There exists a constant $C_0$ such that if $d \geq C_0$, then $G(n,p)$ is almost $d$-regular with high probability.

$$\Pr\left\{ \forall i : 0.9d \leq \deg(i) \leq 1.1d \right\} \geq 0.9.$$

**Proof:**

By Chernoff's inequality,

$$\Pr\left( \deg(i) - d \geq 0.1d \right) \leq \exp\left( -\frac{0.1^2 d}{6} \right) \leq \exp\left( -\frac{0.225 d}{6} \right) \leq \frac{1}{10N} \quad \text{if we choose } C \text{ a large constant.}$$

By union bound:

$$\Pr\left( \bigcup_{i=1}^{n} E_i^c \right) \leq \sum_{i=1}^{n} \Pr(E_i^c) \leq n \cdot \frac{1}{10N} = \frac{1}{10}.$$ 

$$\Rightarrow \Pr\left( \bigcap_{i=1}^{n} E_i \right) \geq 1 - \frac{1}{10} = \frac{9}{10} \quad \text{(de Morgan law)}.$$ 

QED