HOMEWORK 4
HDP KNU+ FALL 2022

Hints are in the back of this homework set.

As in the two previous homeworks, $C, C_1, C_2, \ldots$ and $c, c_1, c_2, \ldots$ denote *positive absolute constants of your choice.* Thus, whenever you see C_1, you can replace it any positive constant you like, for example 10 or 100. Usually, you will find it helpful to choose either big values, like $C_1 = 100$, or small values, like $c = 1/100$, depending on the context. Ask yourself: is making C larger or smaller makes the statement easier to prove? Remember that these constants may not depend on anything, so $C_2 = \sqrt{n}$ is not a valid choice, for example.

In Lecture 9, we proved that, with high probabiliity, sparse random graphs are highly irregular. Namely, we showed that with high probability, the Erdös-Rényi random graph $G(N, p)$ has at least one vertex with disproportionally high degree, say 10 times higher than the expected degree $d = (N-1)p$. The proof, however, had a fault: we assumed that the degrees of different vertices are independent. Fix the fault and give a complete proof of the irregularity theorem:

PROBLEM 1 (IRREGULARITY OF SPARSE RANDOM GRAPHS)

Consider a random graph $G(N, p)$ whose expected degree satisfies $d = (N-1)p < c \log N$. Show that with probability at least 0.9, the exists at least one vertex with degree at least $10d$.

Although from Problem 1 we know that sparse random graphs can have disproportionally high degrees, they can't be insanely high: all degrees are still $O(\log N)$:

PROBLEM 2 (NO TOO HIGH DEGREES)

(a) Check that the classical form of Chernoff's inequality (Theorem 2.3.1 in the book) implies the following simple and useful bound:

$$\mathbb{P}\{S \geq t\} \leq e^{-t} \quad \text{for any } t \geq e^2 \mu,$$

where S is any binomial random variable with mean μ.

(b) Consider a random graph $G(N, p)$ whose expected degree satisfies $d = (N-1)p \leq \log N$. Show that with probability at least 0.9, the following holds: the degrees of all vertices are bounded by $C \log N$.

In Lecture 9, we stated a discrepancy theorem for rectangles in the square $[0,1]^2$. However, in Lecture 10 we proved a version of tis theorem in dimension 1, that is for intervals in $[0,1]$. Prove the original version:

Problem 3 (Discrepancy)

Put N random points into the unit square $[0,1]^2$ independently and according to the uniform distribution. Show that with probability at least 0.99, this set of points is *good* in the following sense.

For any axis-aligned rectangle $I \subset [0,1]^2$ we have

$$\lambda_I N - C\sqrt{\lambda_I N \log N} \le N_I \le \lambda_I N + C\sqrt{\lambda_I N \log N}$$

as long as the left hand side is nonnegative and $N \ge C_1$.

Here N_I denotes the number of points in the rectangle I, λ_I denotes the area of I, and C is an absolute constant.

TURN OVER FOR HINTS

Hints

Hint for Problem 1. As we noted in Lecture 9, it was wrong to say that the degrees $\deg(i)$ and $\deg(j)$ are independent because of the possible edge connecting the vertices i and j. Try to remove that obstacle as follows.

Split the set of N vertices into two subsets A and B, each having $N/2$ vertices. (Assume N is even for simplicity.) For each vertex $i \in A$, define the *quasi-degree* $\deg'(i)$ to be the number of edges connecting i to the vertices in B.

Next, check that (a) the quasi-degrees are independent; (b) the quasi-degrees are bounded above by the true degrees; (c) there exists at least one vertex $i \in A$ with a disproportionally large quasi-degree – this follows by modifying the proof in Lecture 9.

Hint for Problem 2(b). Modify the proof of regularity of dense random graphs, which we gave in Lecture 8. (The same proof appears in the book, see Theorem 2.4.1.) Instead of the small-deviations version of Chernoff's inequality, use the version from part (a).

Hint for Problem 3. Modify the proof of the discrepancy theorem in dimension 1 that we gave in Lecture 10.