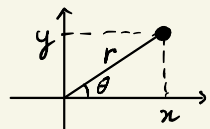
STANDARD NORMAL DISTRIBUTION

- $X \sim N(0,1)$ if $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$

- Why $\int_{-\infty}^{\infty} f(x)\,dx = 1$? $\quad I := \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$?

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right) = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy \quad \text{(Fubini)}$$

Pass to polar coordinates:



$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$I^2 = \underbrace{\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta} = 2\pi \int_0^{\infty} e^{-r^2/2}\, r\,dr.$$

$u = r^2/2$

$du = r\,dr$

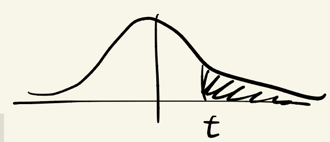
$$= 2\pi \int_0^{\infty} e^{-u}\,du = 2\pi \quad \Rightarrow \quad I = \sqrt{2\pi}$$

- $\mathbb{E}X = \int_{-\infty}^{\infty} \underbrace{x \underbrace{f(x)}_{\text{even}}}_{\text{odd}}\,dx = 0.$

CDF of $N(0,1)$

$\Phi(t) = P\{X \le t\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\,dx$ is a special function.

Approximation:



Lem (Gaussian tail) Let $X \sim N(0,1)$. Then $\forall\, t > 0$:

$$\left(\frac{1}{t} - \frac{1}{t^3}\right)\frac{1}{\sqrt{2\pi}} e^{-t^2/2} \le P\{X \ge t\} \le \frac{1}{t\sqrt{2\pi}} e^{-t^2/2}$$

Upper bound: $P\{X \ge t\} = \frac{1}{2\pi}\int_t^{\infty} e^{-x^2/2}\,dx \le \frac{1}{2\pi}\int_t^{\infty} (x/t)\, e^{-x^2/2}\,dx$, integrate by parts $(du = x\,dx)$

Lower bound: $= \int_t^{\infty} \frac{1}{x}\left(x e^{-x^2/2}\right)dx$, integrate by parts and use this trick

(Chatterjee)

More precise approx: Mills ratio $\frac{P\{X \ge t\}}{f(x)} = \frac{1}{t} - \frac{1}{t^3} + \cdots$ (Taylor series)

pdf

VARIANCE

Def Let X be a r.v. with $\mathbb{E}X = \mu$.
Variance: $Var(X) = \mathbb{E}\left[(X-\mu)^2\right]$ $= \mathbb{E}\left[X^2 - 2\mu X + X^2\right] = \mathbb{E}(X^2) - \mu^2$
Standard deviation: $\sigma(X) := \sqrt{Var(X)}$.

$$\boxed{Var(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2}$$

Ex (a) $X \sim Unif[0,1]$ $\quad \mathbb{E}X = \frac{1}{2}$, $\mathbb{E}[X^2] = \int_0^1 x^2\,dx = \frac{1}{3}$ $\Rightarrow Var(X) = \frac{1}{2} - \left(\frac{1}{3}\right)^2 = \frac{1}{12}$

$\sigma(X) = \frac{1}{\sqrt{12}} \approx 0.29$.

(b) $X \sim Ber(p)$ $\quad \mathbb{E}X = p$, $\mathbb{E}[\underset{X}{\underset{\parallel}{X^2}}] = p$ $\Rightarrow Var(X) = p - p^2 = \boxed{p(1-p)}$

(c) $X \sim N(0,1)$ $\quad \mathbb{E}X = 0$, $\mathbb{E}[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx = \boxed{1}$

$u = x$, $du = xe^{-x^2/2}$

Prop (Properties of variance)

1. $Var(\underset{\text{const}}{\underset{\parallel}{c}}) = 0$
2. $Var(X + c) = Var(X)$
3. $Var(cX) = c^2 \cdot Var(X)$

[straightforward from def]

WARNING:
$Var(X+Y) \neq Var(X) + Var(Y)$ in general

(e.g. for $Y = -X$)

HW: does there exist a r.v. X such that
$X \geq 0$ a.s.
$\mathbb{E}X = 1$
$Var(X) = \infty$?

Ex. (General normal distribution) let $Z \sim N(0,1)$; $\mu \in \mathbb{R}$, $\sigma > 0$,

$$X := \mu + \sigma Z.$$

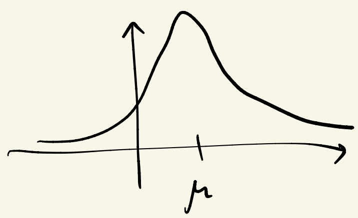
Then we say $X \sim N(\underset{\mathbb{E}X = \mu}{\mu}, \underset{Var(X) = \sigma^2}{\sigma^2})$ (by properties)

Density of $X \sim N(\mu, \sigma^2)$? CDF:

$$F_X(x) = \mathbb{P}\{X \le x\} = \mathbb{P}\{\mu + \sigma Z \le x\} = \mathbb{P}\{Z \le \frac{x-\mu}{\sigma}\}$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$f_X(x) = F_X'(x) = \frac{1}{\sigma}\Phi'\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) = \boxed{\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}}.$$



Cor. to Gaussian tail Bound (p. 28): if $g \sim N(\mu, \sigma^2)$ then

$$\boxed{\mathbb{P}\{|X-\mu| \ge t\sigma\} \le \frac{1}{t\sqrt{2\pi}} e^{-t^2/2} \quad \forall t > 0} \qquad (*)$$

(since $\frac{X-\mu}{\sigma} \sim N(0,1)$).

Ex: $(*) \Rightarrow \mathbb{P}\{|X-\mu| \le 3\sigma\} \ge 0.9970$

while the exact computation gives $= 0.9973$. ☺

Ex (Derangements)

N exams are returned to N students at random.

Expected $\underbrace{\# \text{ of students who receive their own exam}}_{X} = \frac{1}{N}$ (before)

What is the variance of X?

(Why do we care? It could be that

$X = \begin{cases} N \text{ with prob } 1/N \\ 0 \text{ with prob } 1-1/N \end{cases}$ or $X = 1$ with prob. 1 Either case, $\mathbb{E}X = 1$

very different)

$$X = \sum_{i=1}^{N} X_i \quad \text{where } X_i = \begin{cases} 1 & \text{if student } i \text{ gets own exam} \\ 0 & \text{otherwise} \end{cases} \sim \mathrm{Ber}\left(\tfrac{1}{N}\right)$$

$$\mathrm{Var}(X) = \underbrace{\mathbb{E}(X^2)}_{?} - (\underbrace{\mathbb{E}X}_{=1})^2$$

$$\mathbb{E}(X^2) = \mathbb{E}\Big[\big(\sum_{i=1}^{N} X_i\big)^2\Big] = \underbrace{\sum_{i=1}^{N}}_{N \text{ terms}} \mathbb{E}(X_i^2) + \underbrace{\sum_{i \neq j}}_{N^2-N \text{ terms}} \mathbb{E}[X_i X_j]$$

$$\big(\sum_{i=1}^{N} X_i\big)\big(\sum_{j=1}^{N} X_j\big) = \sum_{i,j=1}^{N} X_i X_j = \sum_{i=1}^{N} X_i^2 + \sum_{i \neq j} X_i X_j$$

$X_i \sim \mathrm{Ber}(1/N)$

$$\mathbb{E}[X_i^2] = 1/N$$

$$X_i X_j = \begin{cases} 1, & \text{if both students } i,j \text{ get own exams} \\ 0, & \text{———} \end{cases} \sim \mathrm{Ber}\left(\frac{1}{N(N-1)}\right)$$

\# ways to select 2 exams from N exams, order matters

$$\Rightarrow \mathbb{E}[X^2] = N \cdot \frac{1}{N} + (N^2 - N) \cdot \frac{1}{N(N-1)} = 2.$$

$$\Rightarrow \mathrm{Var}(X) = 2 - 1 = 1. \qquad \Rightarrow \sigma(X) = \sqrt{1} = 1$$

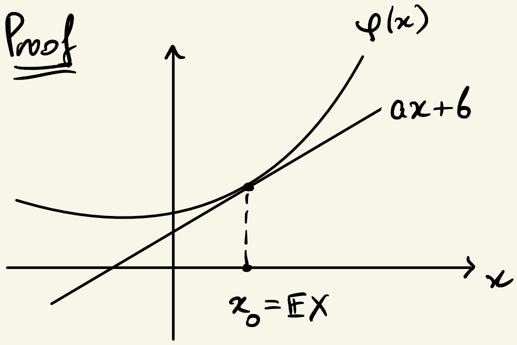
Ex (General normal distribution)

INEQUALITIES

THM (Jensen's inequality) If $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function then

$$\varphi(\mathbb{E}X) \leq \mathbb{E}\varphi(X)$$

Proof



Let $ax+b$ be a tangent to $\varphi(x)$ at $x_0 = \mathbb{E}X$

$\Rightarrow \quad \varphi(x) \geq ax+b \quad \forall x \in \mathbb{R}$.

Substitute $x = X$, take expectation $\Rightarrow$

$\mathbb{E}\varphi(X) \geq a\,\mathbb{E}X + b = ax_0 + b = \varphi(x_0) = \varphi(\mathbb{E}X)$. $\square$