⑥ CONNECTIVITY OF RANDOM GRAPHS

· Erdös-Rényi model $G \sim G(n,p)$: fix n vertices, connect each pair independently with prob. p

THM [ER '1960] Fix $\varepsilon > 0$, let $p_n \in (0,1)$ be a sequence. $G_n \sim G(n, p_n)$, $n \to \infty$

(i) If $p_n > (1+\varepsilon)\frac{\ln n}{n}$, G_n is connected with probability $1 - o(1)$

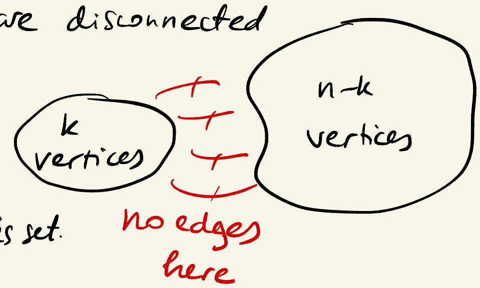
(ii) If $p_n < (1-\varepsilon)\frac{\ln n}{n}$, G_n is disconnected with probability $1 - o(1)$

Remark: Expected degree of $\forall$ vertex $= (n-1)p$.

"Having $\ln n$ friends makes the world connected"

(U.S. Population is $n = 440{,}000{,}000$. $\ln n = 20$)

Proof ⓘ G is disconnected $\Leftrightarrow$ $\exists$ some $k \le \frac{n}{2}$ vertices that are disconnected from the other $n-k$



· The prob. that this happens for a given set of k vertices is $(1-p)^{k(n-k)}$, and there are $\binom{n}{k}$ ways to choose this set.

$$\overset{\text{U.B.}}{\Rightarrow} \quad P\{G \text{ is disconnected}\} \le \sum_{k=1}^{n/2} \binom{n}{k}(1-p)^{k(n-k)} =: a_{n,k} \le ?$$

Use $\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k \quad \forall\ 1 \le k \le n$ **(HW)** $\Rightarrow$

$$a_{n,k} \le \left[\frac{en}{k}(1-p)^{n-k}\right]^k \underset{1-p \le e^{-p}}{\le} \left[\frac{en}{k}e^{-p(n-k)}\right]^k \underset{p > (1+\varepsilon)\frac{\ln n}{n}}{<} \left[\frac{en}{k} n^{-(1+\varepsilon)(1-k/n)}\right]^k.$$

① If $\frac{k}{n} < \frac{\varepsilon}{3}$ then $(1+\varepsilon)(1-k/n) \ge (1+\varepsilon)(1-\varepsilon/3) \ge 1 + \varepsilon/3 \Rightarrow$

$$a_{n,k} < \left[en \cdot n^{-(1+\varepsilon/3)}\right]^k \le \left[en^{-\varepsilon/3}\right]^k.$$

② If $\frac{\varepsilon}{3} \le \frac{k}{n} \le \frac{1}{2}$ then $a_{k,n} \le \left[e \cdot \frac{3}{\varepsilon} n^{-(1+\varepsilon)(1-1/2)}\right]^k \le \left(\frac{3e}{\varepsilon} n^{-1/2}\right)^k$

$$\Rightarrow P\{G \text{ is disconnected}\} \le \sum_{k=1}^{\infty}\left(en^{-\varepsilon/3}\right)^k + \sum_{k=1}^{\infty}\left(\frac{3e}{\varepsilon}n^{-1/2}\right)^k \to 0 \text{ as } n \to \infty$$

(while $\varepsilon > 0$ is fixed). □

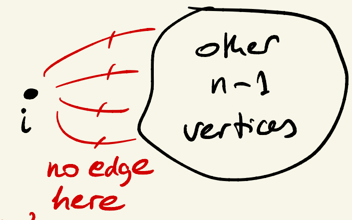
(ii) We will show that G has an <u>isolated vertex</u> with prob $1-o(1)$.

"Second moment method"

$X := \#(\text{isolated vertices})$ WTS: $P\{X>0\} = 1-o(1)$

$$\overset{\Vert}{\sum_{i=1}^{n}} \mathbb{1}\{i \text{ is isolated}\}$$

$$\mu := \mathbb{E}X = \sum_{i=1}^{n} P\{\text{vertex } i \text{ is isolated}\} = n(1-p)^{n-1}$$



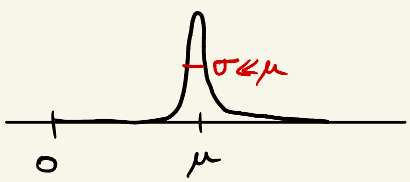
$$(*) \qquad \boxed{= n(1-p)(1-p)^n} \approx \frac{n}{1-p}\cdot e^{-pn}$$

make this rigorous using $1-p \geq e^{-p-p^2}$ $\forall\, p\in(0,1/2)$

$$(**) \qquad \geq \frac{n}{1-p}\cdot e^{-(1-\varepsilon)\ln n} = \frac{n^{\varepsilon}}{1-p} \to \infty \qquad (n\to\infty)$$

- Is this enough to conclude that $X>0$ w.h.p? No.

- But if $\sigma = \sqrt{\mathrm{Var}(X)} \ll \mu$, then enough:

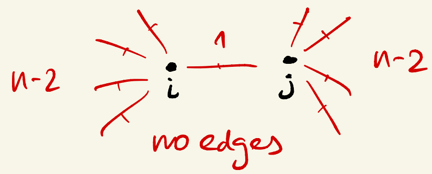


$$P\{X=0\} \leq P\left\{|X-\mu| > \frac{\mu}{2}\right\} \overset{\text{chebyshev}}{\leq} \frac{4\sigma^2}{\mu^2} = 4\cdot\frac{\mathbb{E}X^2-\mu^2}{\mu^2} = 4\left(\frac{\mathbb{E}[X^2]}{(\mathbb{E}X)^2}-1\right) = o(1)$$

if we can show that $\boxed{\dfrac{\mathbb{E}X^2}{(\mathbb{E}X)^2} = 1+o(1)}$

- $$X^2 = \left(\sum_{i=1}^{n}\mathbb{1}_{\{i \text{ is isolated}\}}\right)^2 = \sum_{i,j=1}^{n}\mathbb{1}_{\{i,j \text{ are isolated}\}} = \overbrace{\sum_{i=1}^{n}\mathbb{1}_{\{i \text{ isolated}\}}}^{X} + \sum_{i\neq j}\mathbb{1}_{\{i,j \text{ isolated}\}}$$

$$\mathbb{E}X^2 = \mu + \underbrace{\sum_{i\neq j}}_{n(n-1)\text{ terms}} \underbrace{P\{i,j \text{ isolated}\}}_{=(1-p)^{2(n-2)+1}}$$



$$= \mu + n(n-1)(1-p)^{2n-3}$$

- $$\frac{\mathbb{E}X^2}{\mu^2} \overset{(*)}{=} \underbrace{\frac{1}{\mu}}_{\overset{(**)}{\to}\, 0} + \underbrace{\frac{n(n-1)(1-p)^{2n-3}}{\left(n(1-p)(1-p)^n\right)^2}}_{\to\, 1 \ (n\to\infty,\ p\to 0)} \to 1, \text{ as required.} \qquad \square$$

<u>Remark</u> $\mathrm{diam}(G_{n,p}) \sim \dfrac{\log n}{\log np}$ if $1 < np < c\log n$