Cor (Product of ∞ many measures)

$\forall$ probability measures $\mu_1, \mu_2, \dots$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\exists$ unique prob. measure ν on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$ such that

$$\nu(B_1 \times B_2 \times \dots \times B_n \times \mathbb{R} \times \mathbb{R} \times \dots) = \mu_1(B_1) \cdots \mu(B_n) \quad \forall\ B_i \in \mathcal{B}(\mathbb{R}) \qquad (*)$$

This ν is called the <u>product</u> of μ_i's, denoted

$$\nu := \mu_1 \times \mu_2 \times \cdots$$

Proof $\nu_n := \mu_1 \times \dots \times \mu_n$ are consistent

K.E.T. $\Rightarrow$ $\exists$ extension ν for which $LHS(*) = \nu_n(B_1 \times \dots \times B_n) = RHS(*)$ $\square$

Def (∞ many indep. r.v's) R.v's $X_1, X_2, \dots$ are <u>independent</u> if $\forall$ finite subset of them is independent.

THM (Existence) $\forall$ prob. measures $\mu_1, \mu_2, \dots$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\exists$ independent r.v's $X_1, X_2, \dots$ with marginal distributions $\mu_1, \mu_2, \dots$

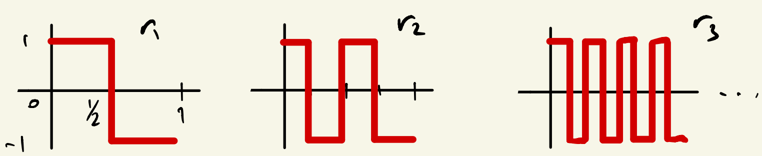
Proof Define $X(\omega) = \omega$ on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \mu_1 \times \mu_2 \times \cdots)$

$$P\{X_1 \in B_1, \dots, X_n \in B_n\} = \underset{\mu_1 \times \dots \times \mu_n}{\underset{\shortparallel}{P}}\{\omega_1 \in B_1, \dots, \omega_n \in B_n\} = \mu_1(B_1) \cdots \mu_n(B_n) = P\{X_1 \in B_1\} \cdots P\{X_n \in B_n\}$$

$\square$

Ex (a) An explicit construction of independent random variables on $([0,1], \mathcal{B}([0,1]), \text{Lebesgue})$

<u>Rademacher functions</u>:



They define independent <u>Rademacher random variables</u> that take values ± 1 with prob $= \frac{1}{2}$ each.

(b) Digits of a random $X \sim \text{Unif}[0,1]$ $\quad 0.1011011101\dots$

$\leftarrow$ HW $\nearrow$

Remark (Uncountable K.E.T.) : $\mathbb{N} \longmapsto T$, an arbitrary set.

- $\mathbb{R}^T$ consists of $\omega = (\omega_t)_{t \in T}$; equivalently, functions $\omega: T \to \mathbb{R}$.
- Cylinder σ-algebra $\sigma(\mathbb{R}^T) = \sigma(\text{cylinders})$, where a cylinder is a set of the form

$$A = \prod_{t \in T} I_t \quad \text{, and each } I_t \text{ is either an interval } (\overset{\overline{\mathbb{R}}}{\overset{\cup}{a_t}}, \overset{\overline{\mathbb{R}}}{\overset{\cup}{b_t}}) \text{ or } \mathbb{R}$$

where all $I_t = \mathbb{R}$ for all but a finite number of the factors.

- For each $n \in \mathbb{N}$ and $t_1, \dots, t_n \in T$, consider a measure $\nu_{t_1, \dots, t_n}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.
- These measures are consistent if

(a) $\nu_{t_1, \dots, t_n}(I_1 \times \dots \times I_n) = \nu_{t_1, \dots, t_n, t_{n+1}}(I_1 \times \dots \times I_n \times \mathbb{R})$

$\forall n \in \mathbb{N}$ and $\forall$ intervals I_i

(b) $\nu_{t_{\pi(1)}, \dots, t_{\pi(n)}}(I_{\pi(1)} \times \dots \times I_{\pi(n)})$

$\forall n \in \mathbb{N}$, $\pi \in S(n)$, intervals I_i

← this extra assumption is needed because T may not be totally ordered (unlike $\mathbb{N}$)
If it is, insist on $t_1 < t_2 < \dots < t_n$ and drop it.

THM (Kolmogorov's Extension Thm – general form)
$\forall$ consistent sequence of probability measures can be uniquely extended to a probability measure ν on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$.

- $\Rightarrow \exists$ uncountably many independent r.v's, e.g.

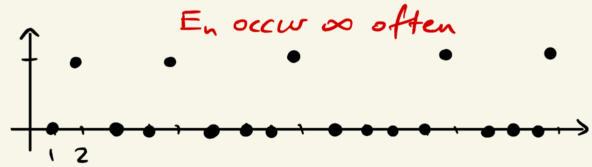
$$(Z_t)_{t \in \mathbb{R}} \quad \text{where all } Z_i \text{ are iid } N(0,1)$$

"White noise"

However, cylinder σ-algebra is coarse (access to finite # of Z_i's)
$\Downarrow$ countable

- $\to$ stochastic processes.

BOREL-CANTELLI LEMMA

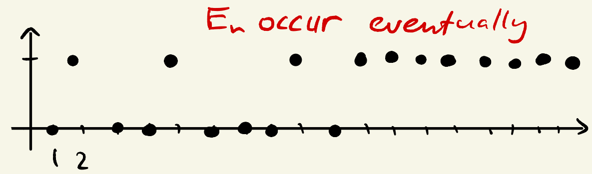


- Consider ∀ events $E_1, E_2, \dots$

$\limsup \mathbb{1}_{E_n} = 1 \iff E_n$ occur infinitely often

$\iff \forall m \in \mathbb{N} \; \exists n \geq m : E_n$ occurs

$\iff \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n$ occurs.



$\liminf \mathbb{1}_{E_n} = 1 \iff E_n$ occur eventually

$\iff \exists m \in \mathbb{N} \; \forall n \geq m : E_n$ occurs

$\iff \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} E_n$ occurs.

Def
- $\limsup E_n := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n :=$ "E_n occur infinitely often"
- $\liminf E_n := \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} E_n :=$ "E_n occur eventually"

$\mathbb{1}_{\limsup E_n} = \limsup \mathbb{1}_{E_n}$; $\qquad \mathbb{1}_{\liminf E_n} = \liminf \mathbb{1}_{E_n}$.

Ex: Fatou lemma : $\mathbb{E}\, \mathbb{1}_{\limsup E_n} \geq \limsup \mathbb{E}\, \mathbb{1}_{E_n}$

$$\mathbb{P}(\limsup E_n) \geq \limsup \mathbb{P}(E_n)$$

Let's upgrade RHS to 1 if E_n are independent:

Borel-Cantelli lemma Let $E_1, E_2, \dots$ be events.

① If $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$ then $\mathbb{P}\{E_n \text{ occur } \infty \text{ often}\} = 0$

② If $\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty$ and E_n are independent

then $\mathbb{P}\{E_n \text{ occur } \infty \text{ often}\} = 1$.

Proof ① By def, $\forall m \in \mathbb{N}$:

$$P(\limsup E_n) \le P\Big(\bigcup_{n=m}^{\infty} E_n\Big) \overset{\text{union bound}}{\le} \sum_{n=m}^{\infty} P(E_n) \to 0 \quad \text{as } m \to \infty.$$

$$\Rightarrow = 0.$$

② $$(\limsup E_n)^c \overset{\text{de Morgan}}{=} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} E_n^c \le \sum_{m=1}^{\infty} \underbrace{P\Big(\bigcap_{n=m}^{\infty} E_n^c\Big)}_{=0}$$

Enough to show that $=0$ $\forall m$ (*)

$$\forall M \ge m: \quad P\Big(\bigcap_{n=m}^{M} E_n^c\Big) \overset{\text{independence}}{=} \prod_{n=m}^{M} \underbrace{P(E_n^c)}_{= 1-P(E_n) \le e^{-P(E_n)}} \qquad (1-x \le e^{-x})$$

$$\le \exp\Big(-\underbrace{\sum_{n=m}^{M} P(E_n)}_{\to\, \infty \text{ as } M\to\infty}\Big) \to 0 \quad \text{as } M \to \infty$$

$\Rightarrow$ (*) follows by continuity of probability □.

Remarks (a) BC 2 may fail for dependent events, e.g. for $E_n = E$.

(b) For independent events, BC implies that

$$P\{E_n \text{ occur } \infty \text{ often}\} \in \{0,1\} \qquad \text{"zero-one law"}$$

More 0/1 laws later.

Cor (Infinite monkey theorem)

A monkey that is hitting keys at random will eventually type the complete works of W. Shakespeare with prob. 1.

(and will retype it ∞ many times)

Proof $E_1 = \{\text{first } N \text{ letters} = \text{works of WS}\}$
$E_2 = \{\text{next } N \text{ letters} = \text{works of WS}\}$
...
independent

$$P(E_n) = p > 0 \;\; \forall n \quad \Rightarrow \quad \sum_n P(E_n) = \infty \overset{BC2}{\Longrightarrow} P(E_n \text{ occur } \infty \text{ often}) = 1. \;\square$$