Example (maximum of gaussians) Let $g_1, g_2, \dots$ be iid $N(0,1)$ r.v's. Then

$$\limsup_n \frac{g_n}{\sqrt{2 \ln n}} = 1 \quad \text{a.s.}$$

Proof The conclusion is equivalent to the following statement. $\forall \varepsilon > 0$:

$$\mathbb{P}\{ g_n > \sqrt{(1+\varepsilon) 2 \ln n} \ \text{ i.o.} \} = 0 \qquad (1)$$

AND

$$\mathbb{P}\{ g_n > \sqrt{(1-\varepsilon) 2 \ln n} \ \text{ i.o.} \} = 1 \qquad (2)$$

To prove this, use gaussian tail bounds (p. 28): if $g \sim N(0,1)$, then

$$\left(\frac{1}{t} - \frac{1}{t^3} \right) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \le \mathbb{P}\{ g > t \} \le \frac{1}{t\sqrt{2\pi}} e^{-t^2/2} \qquad \forall t > 0$$

In particular, $\forall \varepsilon > 0 \ \exists \ t(\varepsilon)$ s.t.:

$$e^{-(1+\frac{\varepsilon}{2}) t^2/2} \le \mathbb{P}\{ g > t \} \le e^{-t^2/2} \qquad \forall \ t > t(\varepsilon)$$

Hence:

- $\mathbb{P}\{ g_n > \sqrt{(1+\varepsilon) 2 \ln n} \} \le \exp\left(-(1+\varepsilon) \ln n \right) = n^{-(1+\varepsilon)}$ is summable

  $\Rightarrow$ (1) follows from Borel-Cantelli I.

- $\mathbb{P}\{ g_n > \sqrt{(1-\varepsilon) 2 \ln n} \} \ge \exp\Big(- \underbrace{(1+\frac{\varepsilon}{2})(1-\varepsilon)}_{\substack{\parallel \\ 1 - \varepsilon/2 - \varepsilon^2/4 \,\le\, 1 - \varepsilon/2}} \ln n \Big) \le n^{-(1-\varepsilon/2)}$ is NOT summable

  $\Rightarrow$ (2) follows from Borel-Cantelli II. $\square$

Example ("Runs") An infinite monkey is hitting keys at random.
How much of William Shakespeare's work will be typed by time n?

Formally: fix an infinite word $W = (W_1, W_2, \dots)$

e.g. $W =$ "TOBEORNOTTOBE..."

and a random word $X = (X_1, X_2, \dots)$ with all $X_i \sim \text{Unif(English alphabet)}$ independent

run of length 2 run of length 4

e.g. $X =$ "ABRATOCADATOBESMTHTOBEORIWANTABANANA..." $L_n = 6$

1 2 3 4 5 6 7 8 $\quad n \quad$ $\ell_6 = 2, \ \ell_7 = 0$

$$\ell_k := \max\{r : X_k = W_k,\ X_{k-1} = W_{k-1}, \dots, X_{k-r+1} = W_{k-r+1}\} \quad \text{(length of a run at time } k\text{)}$$

$$L_n := \max_{1 \le k \le n} \ell_k \quad \text{(max length of a run by time } n\text{)}$$

THM $\dfrac{L_n}{\log_{26} n} \to 1$ a.s. $\quad$ (26 = size of English alphabet)

Proof $\boxed{\text{Upper bound}}$:

$$\mathbb{P}\{\ell_n \ge r\} = \mathbb{P}\{r \text{ last letters are correct}\} = \frac{1}{26^r}$$

$$\Rightarrow \mathbb{P}\{\ell_n > (1+\varepsilon)\log_{26} n\} \le \frac{1}{26^{(1+\varepsilon)\log_{26} n}} = \frac{1}{n^{1+\varepsilon}} \text{ is summable}$$

$\Rightarrow$ by Borel-Cantelli I,

$$\mathbb{P}\{\ell_n > (1+\varepsilon)\log_{26} n \text{ i.o.}\} = 0.$$

$\Rightarrow$ with probability 1,

possibly random

$$\left.\begin{array}{ll} \exists\, m \in \mathbb{N} \ \ \forall n \ge m: & \ell_n < (1+\varepsilon)\log_{26} n. \\ \text{Moreover, } \forall n < m: & \ell_n \le n < m \text{ trivially} \end{array}\right\} \Rightarrow \ell_n < (1+\varepsilon)\log_{26} n + m$$

$$\Rightarrow L_n \le (1+\varepsilon)\log_{26} n + m$$

$$\Rightarrow \frac{L_n}{\log_{26} n} \le 1 + \varepsilon + \frac{m}{\log_{26} n} \quad \frac{m}{\log_{26} n} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \limsup_n \frac{L_n}{\log_{26} n} \le 1 + \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \boxed{\limsup_n \frac{L_n}{\log_{26} n} \le 1 \text{ a.s.}} \ \ ☺$$

$\boxed{\text{Lower Bound}}$:

Partition $\{1,\dots,n\}$ into $\frac{n}{(1-\varepsilon)\log_{26} n}$ blocks of length $(1-\varepsilon)\log_{26} n$. Then

(if some block is a run, $L_n > (1-\varepsilon)\log_{26} n$)

$$\mathbb{P}\left\{\frac{L_n}{\log_{26} n} < 1-\varepsilon\right\} \leq \mathbb{P}\left\{\begin{array}{c}\text{none of the blocks} \\ \text{is a run}\end{array}\right\}$$

$$\leq \Big(\underbrace{1 - \frac{1}{26^{(1-\varepsilon)\log_{26} n}}}_{\text{Prob (given block = run)}}\Big)^{\frac{n}{(1-\varepsilon)\log_{26} n}} \quad \leftarrow \#\text{ blocks (independent)}$$

$$= \left(1 - \frac{1}{n^{1-\varepsilon}}\right)^{\frac{n}{(1-\varepsilon)\log_{26} n}}$$

$$\leq \exp\left(-\frac{n}{n^{1-\varepsilon}(1-\varepsilon)\log_{26} n}\right) \qquad (1-x \leq e^{-x})$$

$$= \exp\left(-\frac{n^{\varepsilon}}{(1-\varepsilon)\log_{26} n}\right) \quad \text{is summable.}$$

$\Rightarrow$ by Borel-Cantelli I,

$$\mathbb{P}\left\{\frac{L_n}{\log_{26} n} < 1-\varepsilon \text{ i.o.}\right\} = 0$$

$\Rightarrow$ with probability 1, $\exists\, m \;\forall\, n \geq m$: $\frac{L_n}{\log_{26} n} \geq 1-\varepsilon$

$$\Rightarrow \liminf \frac{L_n}{\log_{26} n} \geq 1-\varepsilon \quad \Rightarrow \quad \boxed{\liminf \frac{L_n}{\log_{26} n} = 1 \text{ a.s.}}$$

☺

□.

Saint Petersburg Paradox II

Consider an ∞ sequence of games in which one
loses \$$2^n$ with prob. $\frac{1}{2^n+1}$,
wins \$1 with prob. $\frac{2^n}{2^n+1}$, $\quad n = 1, 2, 3, \ldots$

- X_n = winnings of n'th game. $\quad \mathbb{E}X_n = -2^n \cdot \frac{1}{2^n+1} + 1 \cdot \frac{2^n}{2^n+1} = 0$

  $X :=$ total winnings $\Rightarrow \mathbb{E}X = \sum_{n=1}^{\infty} \mathbb{E}X_n = 0 \qquad (*)$

- $\mathbb{P}$(losses occur ∞ often) $= 0$ by Borel-Cantelli

  ($E_n :=$ "n-th game is lost" $\Rightarrow \sum_{n=1}^{\infty} \underbrace{P(E_n)}_{= \frac{1}{2^n+1}} < \infty$)

Hence: with probability 1, starting from some game we will keep winning and will never lose again $\Rightarrow$ winnings $= \infty$

But our expected winnings $= 0$!?

Resolution of paradox: $\mathbb{E}X$ does not exist;

the limit in $(*)$ is not justified

Example ($\sim$ E. Stein's covering lemma)

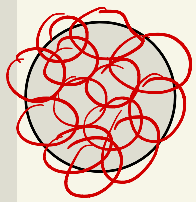
THM let $A_1, A_2, \ldots \subset S^{n-1}$ be any measurable subsets such that

$$\sum_{i=1}^{\infty} \mu(A_i) = \infty.$$

"surface area"

Then $\exists\ U_1, U_2, \ldots \in O(n)$ such that μ-almost every point $x \in S^{n-1}$ belongs to ∞ many sets $U_i A_i$.

$U_i A_i$



Proof let $U_1, U_2, \ldots \in \mathrm{Unif}(O(n))$ and $X \sim \mathrm{Unif}(S^{n-1})$, all independent.

Haar measure $\qquad$ μ, prob. meas. WLOG

Then $U_1^{-1}X, U_2^{-1}X, \ldots \in \mathrm{Unif}(S^{n-1})$ independent (check!) $\quad (*)$

Consider events

$$E_i := \{U_i^{-1} X \in A_i\} = \{X \in U_i A_i\}$$

Note that E_i are independent & $P(E_i) = \mu(A_i)$ by $(*)$.

By assumption, $\sum_{i=1}^{\infty} P(E_i) = \infty$.

Borel-Cantelli II $\Rightarrow$

$$1 = P\{E_i \text{ occur i.o.}\} = \mathbb{E}\,\underbrace{\mathbb{1}_{\{X \in U_i A_i \text{ i.o.}\}}}_{\leq 1}$$

$$\Rightarrow \mathbb{1}_{\{X \in U_i A_i \text{ i.o.}\}} = 1 \text{ a.s.} \quad (**)$$

w.r.to U_i's and X

$\square$

Fubini
$\Rightarrow\ \exists\ U_i : (**)$ holds a.s. w.r.to X (check!)