Zero-One Law

- Borel-Cantelli $\Rightarrow$ $\forall$ independent events $E_1, E_2, \dots$ satisfy
  $\mathbb{P}\{E_n \text{ occur i.o.}\} \in \{0,1\}$.

- General framework:

  Let $X_1, X_2, \dots$ be independent random variables.

  Recall (p. 37-39):

  (a) $\sigma(X_1, \dots, X_n)$ is generated by events $\{X_1 \in B_1, \dots, X_n \in B_n\}$ where $B_i \in \mathcal{B}(\mathbb{R})$

  (b) $\sigma(X_1, X_2, \dots)$ is generated by events $\{X_1 \in B_1, X_2 \in B_2, \dots\}$ where $B_i \in \mathcal{B}(\mathbb{R})$ and $B_i = \mathbb{R}$ for all but finitely many i

  (i.e. events $X^{-1}(B)$ where B = cylinder set in $\mathbb{R}^{\mathbb{N}}$)

Def The tail σ-algebra is

$$\tau(X_1, \dots, X_n) := \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \dots)$$

(σ-algebra indeed)

i.e. $\tau = \tau(X_1, \dots, X_n)$ consists of events that are unaffected by finitely many X_i's

Examples of tail σ-algebras:

(a) $\{X_n \in B_n \text{ i.o.}\}$ $\quad \forall B_n \in \mathcal{B}(\mathbb{R})$

$$\left(= \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{m \geq n} \{X_m \in B_m\}}_{\in\, \sigma(X_n, X_{n+1}, \dots)} \right)$$

(b) $\{\lim X_n = c\}$ $\quad \forall c \in \mathbb{R} \cup \{\pm\infty\}$

$$\left(= \bigcap_{n=1}^{\infty} \{\lim (X_n, X_{n+1}, X_{n+2} \dots) = c\} \right)$$

(c) $\{X_n \text{ converges}\}$ $\quad$ (d) $\{X_n \text{ is bounded}\}$ $\quad$ (e) $\{\sum X_n \text{ converges}\}$

(f) $\left\{ \frac{X_1 + \dots + X_n}{n} \text{ converges} \right\}$

But: $\{\sup X_n \leq 1\}$ and $\{\limsup X_n \leq 1\}$ are NOT tail σ-algebras.

THM (Kolmogorov's zero-one law)

Let $X_1, X_2, \ldots$ be independent random variables.

Then every event $E \in \tau(X_1, X_2, \ldots)$ satisfies $\mathbb{P}(E) \in \{0,1\}$.

Proof. It suffices to show that $\boxed{T \perp\!\!\!\perp T}$ $\forall\, T \in \tau(X_1, X_2, \ldots)$

$\big(\mathbb{P}(T \cap T) = \mathbb{P}(T) \cdot \mathbb{P}(T) \Rightarrow \mathbb{P}(T) \in \{0,1\}$ as claimed.$\big)$

- X_i are independent and $T \in \sigma(X_{n+1}, X_{n+2}, \ldots) \Rightarrow$

$$T \perp\!\!\!\perp \sigma(X_1, \ldots, X_n) \quad \forall n$$

(see p. 39: these two σ-algebras are independent)

$$\Rightarrow T \perp\!\!\!\perp \underbrace{\bigcup_{n=1}^{\infty} \sigma(X_1, \ldots, X_n)}_{=: \mathcal{A}}$$

$\mathcal{A}$ is a π-system

$\ulcorner E, F \in \mathcal{A} \Rightarrow E, F \in \sigma(X_1, \ldots, X_n)$ for some n

$\Rightarrow E \cap F \in \sigma(X_1, \ldots, X_n) \subset \mathcal{A}$ $\lrcorner$

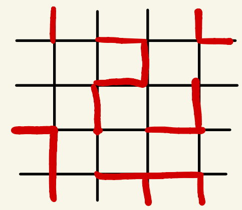
$$\Rightarrow T \perp\!\!\!\perp \sigma(\mathcal{A})$$

(Stability of independence: Thm p. 36)

$\sigma(\mathcal{A}) \overset{\text{def}}{=} \sigma(X_1, X_2, \ldots)$. But $T \in \sigma(X_1, X_2, \ldots) \Rightarrow T \perp\!\!\!\perp T$. $\square$

Examples: (a) All examples on p. 73

(b) Bond percolation on $\mathbb{Z}^2$:



keep each edge with prop. p (remove with prob. $1-p$) independently.

- $E_p = \{\exists$ an infinite connected component$\}$

  $E_p \in \tau(X_{ij} : i,j \in \mathbb{Z})$ where $X_{ij} = \begin{cases} 1 & \text{if } (ij) \text{ is an edge} \\ 0 & \text{if not} \end{cases}$

  ↑ arranged in $\forall$ order.

  Hence $\mathbb{P}(E_p) \in \{0,1\}$.

- $\mathbb{P}(E_p)$ is monotone in p $\Rightarrow$ $\exists$ critical $p_c := \inf\{p : \mathbb{P}(E) = 1\}$

  $p > p_c$: $\exists$ ∞ connected component a.s.

  $p < p_c$: $\nexists$ ∞ connected component a.s.

THM [Kesten '1982]: $p_c = \frac{1}{2}$ (And for $p = p_c$, $\nexists$ ∞ connected component a.s.) ← I think

Open ↑ for $\mathbb{Z}^3$.

(c) Simple random walk ($S_n = X_1 + \dots + X_n$ with iid Rademacher X_i) is called "recurrent" if $S_n = k$ i.o. $\forall k \in \mathbb{Z}$, otherwise called "transient".

0-1 law $\Rightarrow$ $\mathbb{P}\{\text{recurrent}\} \in \{0,1\}$.

Actually, this probability $= 1$.

HW: deduce $\mathbb{P}(\text{recurrent}) = 1$.
For that, prove $\mathbb{P}(\limsup S_n = \infty) = 1$
$\Downarrow$
$\mathbb{P}(\exists n : S_n = k) = 1$

THM $\mathbb{P}\{S_n = 0 \text{ i.o.}\} = 1$

Proof $N := \#(\text{returns to } 0) = \#\{n : S_{2n} = 0\} = \sum_{n=1}^{\infty} \mathbb{1}\{S_{2n} = 0\}$

$$\Rightarrow \mathbb{E}N = \sum_{n=1}^{\infty} \mathbb{P}\{S_{2n} = 0\} = \sum_{n=1}^{\infty} \underbrace{\frac{1}{2^{2n}} \binom{2n}{n}}_{\sim 1/\sqrt{n}} = \infty$$

$\|$ integrated tail (HW 1)

$$\sum_{n=1}^{\infty} \underbrace{\mathbb{P}\{N \ge n\}}_{=\mathbb{P}\{\text{returns; returns; \dots returns}\}} = \sum_{n=1}^{\infty} p^n < \infty \quad \text{if } p < 1$$

$$\mathbb{P}\{\underbrace{\text{returns; returns; \dots returns}}_{n \text{ times}}\} = (\mathbb{P}\{\text{returns}\})^n =: p^n$$

$\Rightarrow p = \mathbb{P}\{\text{ever returns}\} = 1.$

$\Downarrow$ repeat

$\mathbb{P}\{\text{returns i.o.}\} = 1.$ $\square$

slightly informal proof (uses conditioning).

More systematic & general treatment $\longrightarrow$ Markov chains.

SUMS OF INDEPENDENT RANDOM VARIABLES

THM Let X,Y be indep. r.v's with densities f_X, f_Y.
Then $X+Y$ has density $f_{X+Y} = f_X * f_Y$

convolution: $(f_X * f_Y)(z) := \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy$

Proof Thm (p.42) $\Rightarrow$ (X,Y) are jointly abs. continuous, and joint density is

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y). \qquad (*)$$

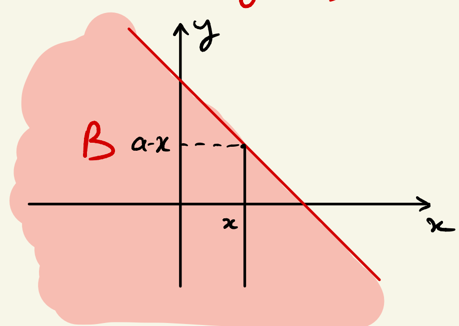
CDF of $X+Y$:

$F_{X+Y}(a) = P\{X+Y \le a\} = P\{(X,Y)\in B\}$ where $B = \{(x,y)\in\mathbb{R}^2 : x+y\le a\}$

$$= \iint_B f_{X,Y}(x,y)\,dx\,dy \ \underset{\text{Fubini}}{\overset{(*)}{=}} \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{a-x} f_X(x) f_Y(y)\,dy}\,dx$$

Change of var: $y = z - x$ $\;\Vert$

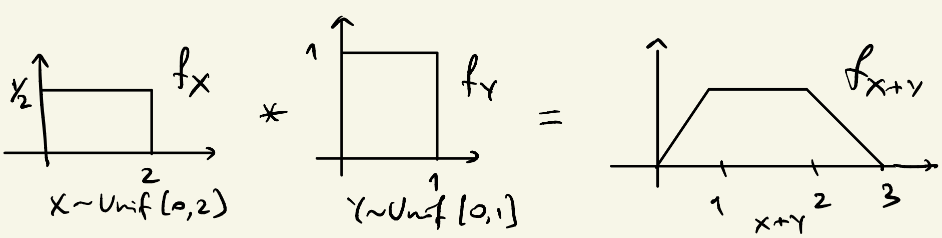
$$\int_{-\infty}^{a} f_X(x) f_Y(z-x)\,dz\,dx$$



$$= \int_{-\infty}^{a} \underbrace{\left[\int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx\right]}\,dz$$

$\Rightarrow$ this must be the density of $X+Y$. $\square$

Examples (a)



(b)