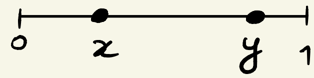
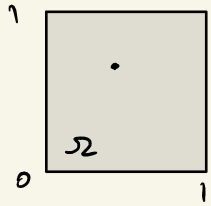
(7) Break a stick at 2 random points:



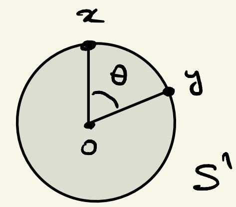
$\Omega = \{(x,y) : x, y \in [0,1]\} = [0,1]^2$

$\mathbb{P}$ = Lebesgue meas (area)

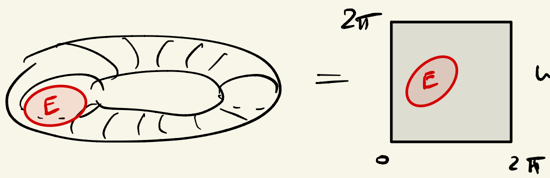
$\mathcal{F}$ = Borel σ-algebra.



(8) Pick 2 random points on a circle



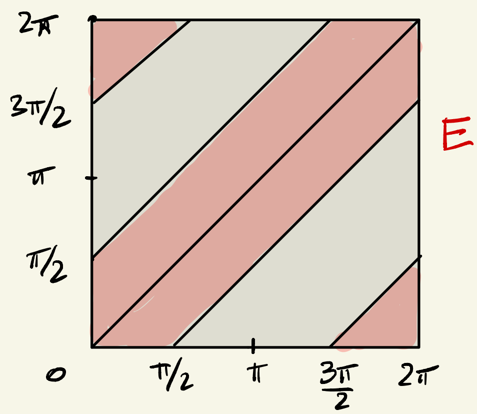
$\Omega = \{(x,y) : x, y \in S^1\} = S^1 \times S^1 = \mathbb{T}^2$ (torus)



with sides glued.

$\mathcal{F}$ = Borel; $\mathbb{P}$ = uniform on $[0,2\pi]^2$, i.e. $\mathbb{P}(E) = \dfrac{\text{Area}(E)}{4\pi^2}$ $\forall E \in \mathcal{F}$

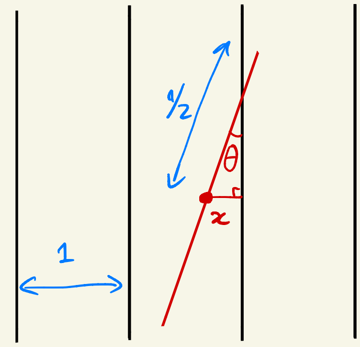
Example: $\mathbb{P}\{x, y \text{ form an acute angle}\} = \mathbb{P}(E) = \dfrac{\text{Area}(E)}{4\pi^2} = \dfrac{1}{2}$



A symmetry proof?

"Geometric probability"

(9) <u>Buffon Needle</u>: "Suppose we have a floor made of parallel strips of wood, each of width 1. Drop a needle of length 1 on the floor. What is the probability that the needle crosses a line between two strips?



<u>Solution</u>: Needle's $\Big(\overset{\mathbb{T}}{\text{direction}}, \overset{\mathbb{T}}{\text{location}}\Big)$ is random, uniformly distributed

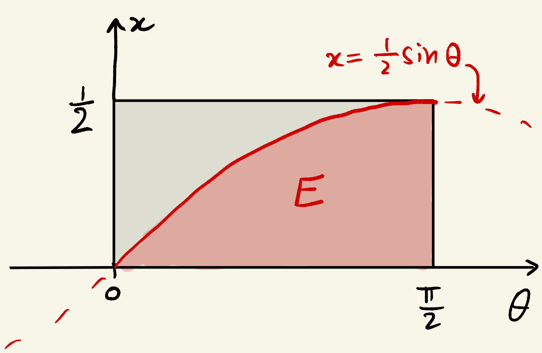
$$\Big\downarrow \text{Parametrize } \mathbb{T}^2$$

$$\left(\begin{array}{l} \theta := \text{acute angle between needle and parallel lines} \\ x := \text{distance from center of needle to closest parallel line} \end{array} \right)$$

$\Omega = \left\{ (\theta, x) : \theta \in [0, \frac{\pi}{2}], \; x \in [0, \frac{1}{2}] \right\}$; $\mathcal{F} =$ Borel σ-algebra.

$\mathcal{F} =$ Borel ; $\mathbb{P} =$ uniform, i.e $\mathbb{P}(E) = \dfrac{\text{Area}(E)}{\pi/4}$

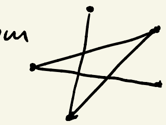
$E :=$ "needle crosses a line" $= \left\{ (\theta, x) \in \Omega : x \leq \frac{1}{2}\sin\theta \right\}$



$$\mathbb{P}(E) = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{2}\sin(\theta)\, d\theta = \boxed{\frac{2}{\pi}}$$

$\Rightarrow$ Computing π empirically ($\longrightarrow$ LLN)

(10) Random graphs. Fix n vertices $\{1,\dots,n\}$, draw M edges at random



$\Omega = \{\text{all graphs on } \{1,\dots,n\} \text{ with } M \text{ edges}\}$

$\mathcal{F} = 2^{\Omega}$

$\mathbb{P}$ = uniform. $\quad |\Omega| = \binom{N}{M}$ where $N = \binom{n}{2}$

$\Rightarrow \forall$ graph $G \in \Omega$, $\quad \mathbb{P}(\{G\}) = \frac{1}{|\Omega|} = \frac{1}{\binom{N}{M}}$

Erdös–Rényi model $G(n, M)$

Examples (a) $\forall$ given vertices i, j,

choose other $M-1$ edges from remaining $N-1$ edges

$$\mathbb{P}(i,j \text{ are connected}) = \frac{\#\{G\in\Omega \text{ in which } i,j \text{ are connected}\}}{\binom{N}{M}} = \frac{\binom{N-1}{M-1}}{\binom{N}{M}} = \frac{M}{N}.$$

(b) $$\mathbb{P}(G \text{ is connected}) \approx \begin{cases} 1 & \text{if } \frac{M}{N} > (1+\varepsilon)\frac{\ln n}{n} \\ 0 & \text{if } \frac{M}{N} < (1-\varepsilon)\frac{\ln n}{n} \end{cases}$$ for $\forall$ fixed $\varepsilon > 0$ and $n \to \infty$

[Erdös–Rényi phase transition]

(11) Random matrices $A = n\times n$ matrix whose all entries are random ± 1:

$\Omega = \{-1, 1\}^{n\times n}$

$\mathcal{F} = 2^{\Omega}$

$\mathbb{P}$ = uniform.

Example: $\mathbb{P}(A \text{ is invertible}) \geq 1 - \left(\frac{1}{2} + o(1)\right)^n$ [K. Tikhomirov 2018]

(12) Haar measure "Random rotations",

$\Omega = O(n)$, orthogonal group

$\mathcal{F}$ = Borel σ-algebra (metric given by Frobenius norm: $\forall\, U, V \in O(n)$: $\|U - V\|_F^2 = \sum_{i,j=1}^{n} (U_{ij} - V_{ij})^2$)

THM [Haar] $\exists$ unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ that is invariant:

$$\mathbb{P}(\mathbf{V}(E)) = \mathbb{P}(E) \quad \forall\, E \in \mathcal{F},\ \mathbf{V} \in O(n).$$

$\mathbb{P}$ is called Haar measure on $O(n)$

Extension: $O(n) \longrightarrow \forall$ locally compact topological group, e.g. $SO(n)$