- Next goal: remove the finite variance assumption from WLLN (p. 79) and thus prove

THM (Khinchin's WLLN) Let $X_1, X_2, \dots$ be iid r.v.'s with finite mean μ. Then $S_n := X_1 + \dots + X_n$ satisfies

$$\frac{S_n}{n} \xrightarrow{P} \mu$$

Proof is based on truncation:

$$\overline{X}_{n,i} := X_i \, \mathbb{1}_{\{|X_i| \le n\}} \; ; \quad \overline{\mu}_n := \mathbb{E}\overline{X}_{n,i} \; ; \quad \overline{S}_n := \overline{X}_{n,1} + \dots + \overline{X}_{n,n}.$$

Approximation:

① (A stronger Markov inequality): $\forall$ r.v. X with finite mean satisfies

$$t \cdot \mathbb{P}\{|X| > t\} \longrightarrow 0 \quad \text{as } t \to \infty$$

$\Big[\; \overset{\|}{\mathbb{E}[t \cdot \mathbb{1}_{\{|X|>t\}}]} \le \mathbb{E}\big[|X| \, \mathbb{1}_{\{|X|>t\}}\big] \to 0$ by DCT

dominated by $|X|$, decreases a.s. to 0 $\;\Big]$

② $\overline{\mu}_n \to \mu$

$\Big[\; \overset{\|}{\mathbb{E}\overline{X}_{n,1}} = \mathbb{E}\Big[\overbrace{X_1 \mathbb{1}_{\{|X_1| \le n\}}}^{X_1 \text{ a.s.}}\Big] \xrightarrow{\text{DCT}} \mathbb{E}X_1 = \mu \;\Big]$

③ $$\begin{aligned} \mathbb{P}\{\overline{S}_n \ne S_n\} &\le \sum_{i=1}^{n} \mathbb{P}\{\overline{X}_{n,i} \ne X_i\} && \text{(union bd.)} \\ &= n \cdot \mathbb{P}\{\overline{X}_{n,1} \ne X_1\} && \text{(identical distr.)} \\ &= n \cdot \mathbb{P}\{|X_1| > n\} && \text{(def. of truncation)} \\ &\to 0 && \text{(stronger Markov inequality ①)} \end{aligned}$$

Next, bound the variance of $\overline{X}_{n,i}$:

④ $\frac{1}{n}\mathbb{E}\left[\overline{X}_{n,1}^2\right] = \frac{1}{n}\int_0^\infty \mathbb{P}\{\overline{X}_{n,1}^2 > u\}\,du$ (integrated tail formula)

$= \frac{2}{n}\int_0^n t\cdot\mathbb{P}\{|\overline{X}_{n,1}| > t\}\,dt$ (change of var. $u = t^2$ & $|\overline{X}_{n,1}| \le n$ everywhere)

$\le \frac{2}{n}\int_0^n \underbrace{t\cdot\mathbb{P}\{|X_1| > t\}}_{\to 0}\,dt$ ($|\overline{X}_{n,1}| \le |X|$ by def)

by stronger Markov ①

$\to 0$ as $n\to\infty$, due to:

Ex (long-term averaging) Let $f: \mathbb{R}^+ \to \mathbb{R}^+$ satisfy

$$f(t) \to 0 \text{ as } t\to\infty.$$

Then

$$\frac{1}{T}\int_0^T f(t)\,dt \to 0 \text{ as } T\to\infty.$$

HW

⑤ Fix $\varepsilon > 0$. WTS: $\mathbb{P}\left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\} \to 0$ as $n\to\infty$.

① $\Rightarrow$ $\exists\, n_0 \in \mathbb{N}$: $|\overline{\mu}_n - \mu| \le \frac{\varepsilon}{2}$ $\forall\, n > n_0$.

$\Rightarrow$ For any such n,

$\mathbb{P}\left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\} \le \mathbb{P}\left\{\left|\frac{S_n}{n} - \overline{\mu}_n\right| > \frac{\varepsilon}{2}\right\}$ (Δ ineq.)

$\le \mathbb{P}\left\{\left|\frac{\overline{S}_n}{n} - \overline{\mu}_n\right| > \frac{\varepsilon}{2}\right\} + \underbrace{\mathbb{P}\{\overline{S}_n \ne S_n\}}_{\to 0}$ by ③

$\mathbb{P}\left\{\left|\frac{\overline{S}_n}{n} - \overline{\mu}_n\right| > \frac{\varepsilon}{2}\right\} \le \frac{\mathrm{Var}(\overline{S}_n/n)}{(\varepsilon/2)^2}$ (Chebyshev)

$= \frac{4}{n^2\varepsilon^2}\mathrm{Var}(\overline{S}_n) = \frac{4}{n\varepsilon^2}\mathrm{Var}(\overline{X}_{n,1})$ (sum of iid r.v.'s)

$\le \frac{4}{n\varepsilon^2}\mathbb{E}\left[\overline{X}_{n,1}^2\right] \longrightarrow 0$ by ④

Q.E.D.