CONVERGENCE IN PROBABILITY VS A.S.

Recall: $X_n \xrightarrow{P} X$ if $\forall \varepsilon>0 \quad \mathbb{P}\{|X_n - X| > \varepsilon\} \to 0$

$X_n \xrightarrow{a.s.} X$ if $\mathbb{P}\{X_n \to X\} = 1$

Lem (Criterion of a.s. convergence)

$$X_n \xrightarrow{a.s.} X \iff \forall \varepsilon>0 \quad \mathbb{P}\{|X_n - X| > \varepsilon \text{ i.o.}\} = 0$$

$\ulcorner$ $\Updownarrow$

$\mathbb{P}\{X_n \not\to X\} = 0$ (WLOG $\varepsilon \in \mathbb{Q}$ $\Rightarrow$ countable union)

$\Updownarrow$

$\mathbb{P}\{\exists\, \varepsilon>0 : |X_n - X| > \varepsilon \text{ i.o.}\} = 0$ $\lrcorner$

THM (Convergence in prob. vs a.s.)

(i) $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X$

(ii) The converse is not always true

(iii) $X_n \xrightarrow{P} X \Rightarrow \exists$ subsequence $X_{n_k} \xrightarrow{a.s.} X$

(iv) $X_n \xrightarrow{P} X$ and (X_n) is monotone a.s. $\Rightarrow X_n \xrightarrow{a.s.} X$

$\ulcorner$ (i) Fix $\varepsilon>0$, let $E_n := \{|X_n - X| > \varepsilon\}$.

$$\limsup \mathbb{P}(E_n) \overset{\text{Fatou}}{\leq} \mathbb{P}(\limsup E_n) \overset{\text{Lem}}{=} 0 \quad \square$$

(ii) Let $X_n \sim \mathrm{Ber}(1/n)$ be independent. Then $X_n \xrightarrow{P} 0$ $\left(\mathbb{P}\{|X_n|>\varepsilon\} = \frac{1}{n} \to 0\right)$

but $X_n \overset{a.s.}{\not\to} 0$ $\left(\mathbb{P}\{X_n = 1 \text{ i.o.}\} = 1\right.$ by Borel-Cantelli II, since $\sum_n \frac{1}{n} = \infty$

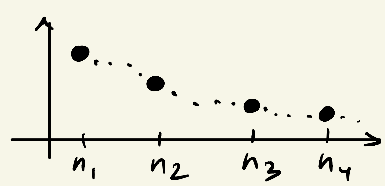
Finish by Lem. above. $\left.\right)$

(iii) Assume $X_n \xrightarrow{P} 0$, let $\varepsilon>0$. Choose a subsequence s.t

$$\mathbb{P}\{|X_{n_k} - X| > \varepsilon\} < \frac{1}{k^2}$$

Thus, by Borel-Cantelli I,

$$\mathbb{P}\{|X_{n_k} - X| > \varepsilon \text{ i.o.}\} = 0.$$



Finish by Lem. above.

(iv) By (iii) $\exists$ subsequence $X_{n_k} \xrightarrow{a.s.} X$. Monotonicity (interpolate) $\Rightarrow$

$X_n \xrightarrow{as} X$. $\lrcorner$

Cor (Convergence in prob. vs a.s.)

$X_n \xrightarrow{P} X \iff \forall$ subsequence $(X_{n_k}) \ \exists$ further subsequence $(X_{n_{k_\ell}})$ s.t. $X_{n_{k_\ell}} \xrightarrow{a.s.} X$.

$\ulcorner$ ($\Rightarrow$) follows from Thm p. 89 (iii) for (X_{n_k}).

($\Leftarrow$) Fix $\varepsilon>0$, $E_n := \{|X_n - X| > \varepsilon\}$, $p_n := \mathbb{P}(E_n) \to 0$?

$\forall (n_k) \ \exists (n_{k_\ell}) : X_{n_{k_\ell}} \xrightarrow{a.s.} X \Rightarrow X_{n_{k_\ell}} \xrightarrow{P} X \Rightarrow p_{n_{k_\ell}} \to 0$ $\lrcorner$

$\Rightarrow p_n \to 0$, due to:

Lemma (Convergence Criterion) A sequence in a topological space converges to $x \iff \forall (n_k) \ \exists (n_{k_\ell}) : x_{n_{k_\ell}} \to x$.

$\ulcorner$ Easy by contradiction: if $x_n \not\to x$ then $\exists V \ni x$, $\exists (n_k) : x_{n_k} \notin V$
open
$\Rightarrow x_{n_{k_\ell}} \not\to x \ \forall (n_{k_\ell})$ $\lrcorner$

Remarks (a) A.S. convergence does NOT come from any topology (otherwise Cor. & Lem. above would yield that $X_n \xrightarrow{P} X \Leftrightarrow X_n \xrightarrow{a.s.} X$)

(b) Convergence of probability is metrizable

HW e.g. by Ky Fan metric

$d_1(X,Y) = \inf\{\varepsilon>0 : \mathbb{P}\{|X-Y|>\varepsilon\} < \varepsilon\}$ or $d_2(X,Y) = \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right]$

Exercise (Continuous Mapping Theorem)

If $X_n \xrightarrow{P} X$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f(X_n) \xrightarrow{P} f(X)$

$\ulcorner$ Cor. above: $\forall (n_k) \ \exists (n_{k\ell}) : X_{n_{k\ell}} \xrightarrow{a.s.} X \Rightarrow f(X_{n_{k\ell}}) \xrightarrow{a.s.} f(X)$ (usual continuity in $\mathbb{R}$) $\Rightarrow f(X_{n_{k\ell}}) \xrightarrow{P} f(X)$. Finish by Cor. above. $\lrcorner$

STRONG LAW OF LARGE NUMBERS

A.S. convergence

- First, under the 4^{th} moment condition: $\mathbb{E}X^4 < \infty$

Recall that 4^{th} moment implies 2^{nd} moment $\left(\|X\|_{L^2} \le \|X\|_{L^4}\right) \Rightarrow$ finite variance.

THM (SLLN under 4^{th} moment) Let $X_1, X_2, \dots$ be iid rv's with $\mathbb{E}X_i = \mu$, $\mathbb{E}X_i^4 =: K < \infty$.

Then $S_n = X_1 + \cdots + X_n$ satisfies

$$\frac{S_n}{n} \xrightarrow{a.s.} \mu.$$

Proof W.L.O.G. $\mu = 0$ (otherwise replace $X_i \mapsto X_i - \mu$).

$$\mathbb{E}[S_n^4] = \mathbb{E}\Big[(X_1+\cdots+X_n)(X_1+\cdots+X_n)(X_1+\cdots+X_n)(X_1+\cdots+X_n)\Big]$$

$$= \mathbb{E}\Big[\underbrace{\sum X_i^4}_{n \text{ terms}} + \sum X_i^3 X_j + \underbrace{\sum X_i^2 X_j^2}_{\binom{n}{2}\binom{4}{2} \text{ terms}} + \sum X_i^2 X_j X_k + \sum X_i X_j X_k X_\ell\Big]$$

Expectation of all those terms equals zero

(choose a pair (i,j), choose 2 factors it comes from)

$$= \underbrace{n \cdot \mathbb{E}[X_1^4]}_{= \, K} + \underbrace{\binom{n}{2}\binom{4}{2}}_{= \, 3n(n-1)} \cdot \underbrace{\mathbb{E}[X_1^2 X_2^2]}_{\le \, (\mathbb{E}X_1^4)^{1/2} (\mathbb{E}X_2^4)^{1/2} \le K}$$

Cauchy-Schwarz

$$\le nK + 3n(n-1)K \le 3Kn^2$$

$$\Rightarrow \mathbb{E}\left[\left(\frac{S_n}{n}\right)^4\right] \le \frac{3K}{n^2}$$

- Markov $\Rightarrow \forall \varepsilon > 0$: $\mathbb{P}\Big\{\overbrace{\left|\tfrac{S_n}{n}\right| > \varepsilon}^{E_n}\Big\} = \mathbb{P}\left\{\left(\frac{S_n}{n}\right)^4 > \varepsilon^4\right\} \le \frac{\mathbb{E}\left[(S_n/n)^4\right]}{\varepsilon^4} \le \frac{3K}{n^2\varepsilon^4}$.
- $\sum_n \mathbb{P}(E_n) < \infty \Rightarrow$ by Borel-Cantelli, $\mathbb{P}\{E_n \text{ occurs i.o.}\} = 0$
- Lem (criterion p. 89) $\Rightarrow S_n/n \xrightarrow{a.s.} 0$ □.

"Cor" Flip a coin continually $\Rightarrow$ with prob. 1, the fraction of heads $\to 1/2$

$\ulcorner X_i = \begin{cases} 1 & \text{if } i\text{'th flip} = H \\ 0 & \text{if } T \end{cases}.\lrcorner$