Next, remove the 4^{th} moment assumption:

THM (Strong Law of Large Numbers) [Kolmogorov 1930]

Let $X_1, X_2, \dots$ be iid r.v's with finite mean $\mathbb{E}X_i = \mu$.

thus, we assume $\mathbb{E}|X_i| < \infty$

Then $S_n := X_1 + \dots + X_n$ satisfies

$$\frac{S_n}{n} \xrightarrow{a.s.} \mu$$

Proof ① (Truncation)

WLOG, $\boxed{X_k \geq 0 \ \forall k}$ (if not, $X_k = X_k^+ - X_k^-$ and handle X_k^+, X_k^- separately)

$$Y_k := X_k \mathbb{1}_{\{X_k \leq k\}}, \qquad T_n := Y_1 + \dots + Y_n.$$

② (Subsequence) We will first prove convergence for a (geometric) subsequence

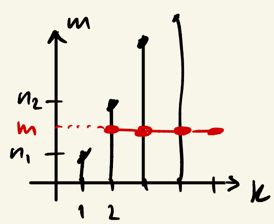
Fix $\alpha > 1$, let $n_k := \lceil \alpha^k \rceil$, $k = 1, 2, \dots$

In preparation for Borel-Cantelli, let's bound

$$\sum_{k=1}^{\infty} \mathbb{P}\left\{ \left| \frac{T_{n_k} - \mathbb{E}T_{n_k}}{n_k} \right| > \varepsilon \right\} \leq \sum_{k=1}^{\infty} \frac{Var(T_{n_k}/n_k)}{\varepsilon^2} \quad \text{(Chebyshev)}$$

$$= \sum_{k=1}^{\infty} \sum_{m=1}^{n_k} \frac{Var(Y_m)}{\varepsilon^2 n_k^2} \overset{\text{Fubini-Tonelli}}{=} \frac{1}{\varepsilon^2} \sum_{m=1}^{\infty} Var(Y_m) \underbrace{\boxed{\sum_{k : n_k \geq m} \frac{1}{n_k^2}}}_{\leq \frac{C(\alpha)}{m^2} \text{ (geometric series)}}$$

$$\leq \frac{C(\alpha)}{\varepsilon^2} \boxed{\sum_{m=1}^{\infty} \frac{Var(Y_m)}{m^2}} \leq ?$$



③ (Variance) $Var(Y_m) \leq \mathbb{E}Y_m^2 = \int_0^\infty \mathbb{P}\{Y_m^2 > y\}\, dy$ (integrated tail)

$$= 2\int_0^\infty x\, \mathbb{P}\{Y_m > x\}\, dx \quad \text{(change of var } y = x^2 \text{)}$$

$$= 2\int_0^m x\, \mathbb{P}\{X_m > x\}\, dx \quad \text{(def of truncation)}$$

$$= 2\int_0^m x\, \mathbb{P}\{X_1 > x\}\, dx \quad \text{(identical distribution)}$$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{\mathrm{Var}(Y_m)}{m^2} = \sum_{m=1}^{\infty} \int_0^{\infty} \frac{2}{m^2} \mathbb{1}_{\{x \le m\}}\, x\, P\{X_1 > x\}\, dx$$

$$= 2 \int_0^{\infty} \underbrace{\left(\sum_{m=1}^{\infty} \frac{\mathbb{1}_{\{x \le m\}}}{m^2} \right)}_{\sum_{m \ge x} \frac{1}{m^2} \asymp \int_x^{\infty} \frac{dm}{m^2} \asymp \frac{1}{x}} x\, P\{X_1 > x\}\, dx \qquad \text{(Fubini–Tonelli)}$$

$$\asymp \int_0^{\infty} P\{X_1 > x\}\, dx = \mathbb{E}X_1 < \infty \qquad \text{(integrated tail)}$$

④ We have shown that

$$\sum_{k=1}^{\infty} P\left\{ \left| \frac{T_{n_k} - \mathbb{E}T_{n_k}}{n_k} \right| > \varepsilon \right\} < \infty \qquad \forall \varepsilon > 0$$

Borel-Cantelli $\Rightarrow$

$$P\left\{ \left| \frac{T_{n_k} - \mathbb{E}T_{n_k}}{n_k} \right| > \varepsilon \text{ i.o.} \right\} = 0 \qquad \forall \varepsilon > 0.$$

Criterion of a.s. convergence (Lem p. 89) $\Rightarrow$

$$\boxed{\frac{T_{n_k} - \mathbb{E}T_{n_k}}{n_k} \to 0 \quad \text{a.s.}} \qquad (*)$$

⑤ (Replacing $\mathbb{E}T_{n_k}$ with μ)

$$\mathbb{E}Y_k = \mathbb{E}\big[\underbrace{X_1 \mathbb{1}_{\{X_1 \le k\}}}_{\downarrow \text{a.s.}\ X_1}\big] \xrightarrow[\text{D.C.T.}]{} \mathbb{E}X_1 = \mu \quad \text{as } k \to \infty$$

$$\Rightarrow \frac{\mathbb{E}T_n}{n} = \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}Y_k \to \mu \quad \text{as } n \to \infty \quad \longrightarrow$$

HW

Fact (convergence of sequences in $\mathbb{R}$)

If $x_n \to x$ then

$$\frac{x_1 + \cdots + x_n}{n} \to x$$

$$\Rightarrow \frac{\mathbb{E}T_{n_k}}{n_k} \to \mu \quad \text{as } k \to \infty.$$

Combining with $(*)$ $\Rightarrow$

$$\boxed{\frac{T_{n_k}}{n_k} \to \mu \quad \text{a.s.}}$$

⑥ (Interpolation) $\forall n \in \mathbb{N} \quad \exists k: \quad n_k \le n \le n_{k+1}$

$$\Rightarrow \quad T_{n_k} \le T_n \le T_{n_{k+1}} \quad \text{(monotone by def.)}$$

$$\Rightarrow \quad \frac{T_{n_k}}{n_{k+1}} \le \frac{T_n}{n} \le \frac{T_{n_{k+1}}}{n_k}$$

$$\frac{T_{n_k}}{n_{k+1}} = \underbrace{\frac{T_{n_k}}{n_k}}_{\substack{⑤ \downarrow a.s. \\ \mu}} \cdot \underbrace{\frac{n_k}{n_{k+1}}}_{\substack{\downarrow \\ 1/\alpha \text{ (def.)}}} \qquad \frac{T_{n_{k+1}}}{n_k} = \underbrace{\frac{T_{n_{k+1}}}{n_{k+1}}}_{\substack{⑤ \downarrow a.s. \\ \mu}} \cdot \underbrace{\frac{n_{k+1}}{n_k}}_{\substack{\downarrow \\ \alpha \text{ (def)}}}$$

$\Rightarrow$ with probability 1,

$$\frac{\mu}{\alpha} \le \frac{\liminf T_n}{n} \le \frac{\limsup T_n}{n} \le \alpha \mu$$

This holds $\forall \alpha > 1 \Rightarrow$ with prob. 1, $\frac{T_n}{n} \to \mu$.

i.e.

$$\frac{T_n}{n} \xrightarrow{a.s.} \mu$$

⑦ (Replacing T_n by S_n):

$$\sum_{k=1}^{\infty} \mathbb{P}\{|X_k| > k\} = \sum_{k=1}^{\infty} \mathbb{P}\{|X_1| > k\} \qquad \text{(identical distr.)}$$

$$= \sum_{k=1}^{\infty} \int_{k-1}^{k} \underbrace{\mathbb{P}\{|X_1| > k\} dt}_{\le \; \mathbb{P}\{|X_1| > t\} dt} \le \int_0^{\infty} \mathbb{P}\{|X_1| > t\} dt = \mathbb{E}|X_1| < \infty$$

(or just apply integral test)

BCI

$\Rightarrow \mathbb{P}\{X_k \ne Y_k \text{ i.o.}\} = 0$

$\Rightarrow$ with prob. 1, $\exists k_0 \quad \forall k > k_0: \quad X_k = Y_k$

$$\Rightarrow \quad |S_n - T_n| \le \sum_{k=1}^{k_0} |X_k| =: R \qquad \forall n$$

$$\Rightarrow \left| \frac{S_n}{n} - \underbrace{\frac{T_n}{n}}_{\downarrow \, \mu} \right| \le \frac{R}{n} \to 0 \quad \text{as } n \to \infty$$

$\Rightarrow S_n/n \to \mu$. $\square$

THM (SLLN for ∞ mean) [Kolmogorov 1933]

Let $X_1, X_2, \dots$ be iid r.v's, $S_n := X_1 + \dots + X_n$

If $\mathbb{E}|X_i| = \infty$ then $\frac{S_n}{n}$ is unbounded (and thus diverges) a.s.

Proof ① $|X_n| = |S_n - S_{n-1}| \le |S_n| + |S_{n-1}|$

$\Rightarrow \limsup_n \frac{|X_n|}{n} \le 2 \limsup_n \frac{|S_n|}{n}$

Hence, it is enough to show that

$$\limsup_n \frac{|X_n|}{n} = \infty \quad \text{a.s.}$$

② $\sum_{n=1}^{\infty} P\{|X_n| > n\} = \sum_{n=1}^{\infty} P\{|X_1| > n\}$ (identical distribution)

$\asymp \int_0^{\infty} P\{|X_1| > t\}\, dt$ (integral test)

$= \mathbb{E}|X_1| = \infty.$ (integrated tail)

③ Let $M > 0$, apply ② for X_n/M (which have ∞ mean, too) $\Rightarrow$

$$\sum_{n=1}^{\infty} P\{|X_n| > Mn\} = \infty$$

Borel-Cantelli II $\Rightarrow$ $P\{\, |\frac{X_n}{n}| > M \text{ i.o.} \} = 1.$

$\Rightarrow P\{\limsup_n |\frac{X_n}{n}| > M\} = 1$

Take intersection over $M = 1, 2, 3, \dots \Rightarrow P$

$$P\{\limsup \frac{|X_n|}{n} = \infty\} = 1. \qquad \square.$$

Ex If $X_i \sim$ Cauchy iid $\Rightarrow \frac{S_n}{n} \sim$ Cauchy.

LLN fails. Does not contradict Thm: a sequence of Cauchy r.v's may be unbounded.