THM [Etemadi's maximal inequality '1985]

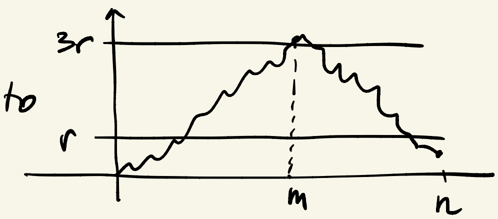
Let $X_1, \dots, X_n$ be independent r.v's, $\quad S_k := X_1 + \dots + X_k$

Then $\forall\, t > 0$:

$$\mathbb{P}\Big\{ \max_{1 \le k \le n} |S_k| \ge \underset{\substack{\shortparallel \\ 3r}}{t} \Big\} \le 3 \max_{1 \le k \le n} \mathbb{P}\Big\{ |S_k| \ge \underset{\substack{\shortparallel \\ r}}{\tfrac{t}{3}} \Big\}$$

Trivial if $X_i \ge 0$; nontrivial otherwise.

Proof Break down the event in LHS according to when $|S_k| \ge 3r$ for the first time



$$\Big\{ \max_{1 \le k \le n} |S_k| \ge 3r \Big\} = \bigsqcup_{m=1}^{n} \underbrace{\Big\{ \max_{1 \le k < m} |S_k| < 3r \text{ AND } |S_m| \ge 3r \Big\}}_{\substack{\shortparallel \\ E_m}}$$

(disjoint events)

Also, consider whether $F_n := \{ |S_n| \ge r \}$ holds:

$$= F_n \cup \bigsqcup_{m=1}^{n} \left(E_m \cap F_n^c \right)$$

$$\Rightarrow \mathbb{P}\Big\{ \max_{1 \le k \le n} |S_k| \ge 3r \Big\} \le \mathbb{P}(F_n) + \sum_{m=1}^{n} \mathbb{P}\big(\underbrace{E_m \cap F_n^c}_{} \big)$$

this event implies $|S_m - S_n| \ge |S_m| - |S_n| \ge 3r - r \ge 2r$

$$\le \mathbb{P}(F_n) + \sum_{m=1}^{n} \mathbb{P}\Big(\underbrace{E_m}_{\substack{\cap \\ \sigma(X_1, \dots, X_m)}} \cap \underbrace{\{ |S_m - S_n| \ge 2r \}}_{\substack{\cap \\ \sigma(X_{m+1}, \dots, X_m)}} \Big)$$

independent

$$= \mathbb{P}(F_n) + \sum_{m=1}^{n} \mathbb{P}(E_m) \cdot \mathbb{P}\{ |S_m - S_n| \ge 2r \}$$

$$\le \mathbb{P}(\underbrace{F_n}_{\substack{\shortparallel \\ \{S_n \ge r\}}}) + \max_{1 \le m \le n} \mathbb{P}\{ \underbrace{|S_m - S_n| \ge 2r}_{} \}$$

this event implies $|S_m| \ge r$ OR $|S_n| \ge r$

$$\le 3 \max_{1 \le k \le n} \mathbb{P}\{ |S_k| \ge r \}. \qquad \square$$

Remark Integrated tail formula $\Rightarrow$ $\mathbb{E} \max_{1 \le k \le n} |S_k| \le 9 \max_{1 \le k \le n} \mathbb{E}|S_k|$

• If $\mathbb{E}X_i = 0$ and $\mathrm{Var}(X_i) < \infty$,

$$\mathrm{RHS(Etemadi)} \le 3 \max_{1\le k\le n} \frac{\mathrm{Var}(S_k)}{(t/3)^2} \qquad \text{(Chebyshev)}$$

$$\le 27\,\mathrm{Var}(S_n) \qquad \left(\text{since } \mathrm{Var}(S_k) = \sum_{i=1}^{k} \mathrm{Var}(X_i) \text{ increases in } k\right)$$

$\Rightarrow$

THM [Kolmogorov's maximal inequality]

Let $X_1, \dots, X_n$ be independent r.v's with $\mathbb{E}X_i = 0$, $\mathrm{Var}(X_i) < \infty$ $\forall i$

Let $S_k := X_1 + \dots + X_k$. Then $\forall\, t \ge 0$:

$$P\left\{ \max_{1\le k\le n} |S_k| \ge t \right\} \le \frac{27}{t^2} \mathrm{Var}(S_n)$$

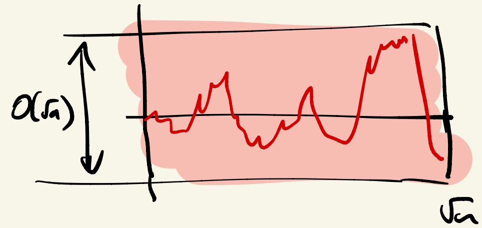
Remark Constant 27 can be improved to 1 (which is obviously sharp) [Kolmogorov]

Example: Simple random walk $P\{X_i = \pm 1\} = \frac{1}{2}$ (Rademacher r.v's)

$$\mathrm{Var}(S_n) = \sum_1^n \mathrm{Var}(X_i) = n$$

$$\text{KMI} \Rightarrow P\left\{ \max_{1\le k\le n} |S_k| \ge t \right\} \le \frac{n}{t^2} \le 0.01 \quad \text{if } t \asymp \sqrt{n}$$

$$\Rightarrow \boxed{\max_{1\le k\le n} |S_k| \asymp \sqrt{n} \text{ whp}}$$



CONVERGENCE OF RANDOM SERIES :

THM [Kolmogorov's two series thm]

Let $X_1, X_2, \dots$ be independent mean zero r.v's.

If $\sum_{n=1}^{\infty} Var(X_n) < \infty$ then $\sum_{n=1}^{\infty} X_n$ converges a.s.

Proof $S_n := X_1 + \cdots + X_n$. Enough to show that, with probability 1,

S_n is Cauchy. $\iff$ $w_p := \sup_{m,n > p} |S_m - S_n| \to 0$ as $p \to \infty$

Fix p, N and apply Kolmogorov's maximal inequality for $X_{p+1}, \dots, X_N$ $\Rightarrow$

$$\Rightarrow \quad P\{ \underbrace{\max_{n \in (p,N]} |X_{p+1} + \cdots + X_n| > \varepsilon}_{\uparrow \text{ increasing events in } N} \} \le \frac{27}{\varepsilon^2} \underbrace{\sum_{k=p+1}^{N} Var(X_k)}_{\text{converges as } N \to \infty} \Rightarrow \text{taking limit,}$$

$$P\{ \sup_{n > p} |S_n - S_p| > \varepsilon \} \le \frac{27}{\varepsilon^2} \underbrace{\sum_{k=p+1}^{\infty} Var(X_k)}_{\downarrow \, 0 \text{ as } n \to \infty} \qquad \forall n \in \mathbb{N}.$$

- By Δ inequality, $w_p \le \sup_{m > p} |S_m - S_p| + \sup_{n > p} |S_n - S_p|$

$\Rightarrow$ if $w_p > 2\varepsilon$ then $\sup_{n > p} |S_n - S_p| > \varepsilon$ $\Rightarrow$

$$P\{ w_p > 2\varepsilon \} \le P\{ \sup_{n > p} |S_n - S_p| > \varepsilon \} \longrightarrow 0 \quad \text{as } p \to \infty$$

i.e. $w_p \longrightarrow 0$ in probability.

- Since w_p is monotonically decreasing in p,

$w_p \xrightarrow{a.s.} 0$ (Thm (iv) p. 89) $\square$

Remarks. The converse is NOT true (HW)

- Kolmogorov's three series thm gives necessary and sufficient conditions for a.s. convergence of series.

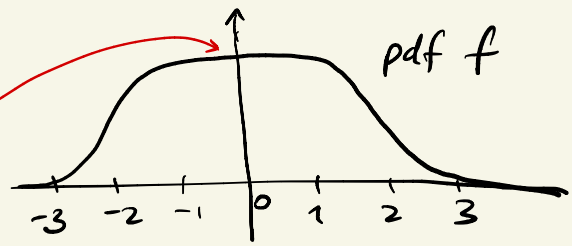
Examples (Random sign series) let $X_n = \pm 1$ with prob $1/2$ (Rademacher) iid.

(a) $\sum_{n=1}^{\infty} \frac{X_n}{2^n}$ converges a.s. limit $\sim$ Unif$[-1,1]$ (HW)

(b) Random harmonic series: $S = \sum_{n=1}^{\infty} \frac{X_n}{n}$ converges a.s. since

$$\sum_{n=1}^{\infty} \mathrm{Var}\left(\frac{X_n}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$\Rightarrow \mathbb{E}S = 0, \quad \mathrm{Var}(S) = \frac{\pi^2}{6}$



X has density;

$$f(0) = \frac{1}{\pi} \int_0^{\infty} \prod_{k=1}^{\infty} \cos\left(\frac{x}{n}\right) dx$$

(can be shown using characteristic functions)

"infinite cosine product integral"

$f(0) = 0.124\underbrace{9999999 \ldots\ldots 9}_{\sim 40 \text{ digits} = 9}764 \neq \frac{1}{4}$! (differs from $1/4$ by $< 10^{-42}$)