EQUIVALENT PROPERTIES:

Recall $X_n \xrightarrow{w} X \overset{\text{def}}{\iff} P\{X_n \le x\} \to P\{X \le x\}$ whenever $P\{X = x\} = 0$.

Lem (Portmanteau lemma) TFAE:

(i) $X_n \xrightarrow{w} X$

(ii) $P\{X_n \in [a,b]\} \longrightarrow P\{X \in [a,b]\}$ whenever $P\{X=a\} = P\{X=b\} = 0$

(iii) $P\{X_n \in B\} \longrightarrow P\{X \in B\}$ $\forall$ continuity set $B \subset \mathbb{R}$ of X

(iv) $\mathbb{E}h(X_n) \longrightarrow \mathbb{E}h(X)$ $\forall$ bounded, continuous $h : \mathbb{R} \to \mathbb{R}$

(v) $\mathbb{E}h(X_n) \longrightarrow \mathbb{E}h(X)$ whenever h and all its derivatives $h, h', h'', \dots$ are bounded & compactly supported.

i.e. $\forall$ set B such that $P\{X \in \partial B\} = 0$ where $\partial B = \overline{B} \setminus B^\circ$ is the boundary of B

Proof

(i) $\Rightarrow$ (iii) By Skorokhod's representation thm (p. 111), we can assume wLOG that $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X$. Fix $\forall \varepsilon > 0$

$$P\{X_n \in B\} \le P\{\underbrace{X_n \in B \;\&\; |X_n - X| < \varepsilon}_{\Downarrow \Delta}\} + \underbrace{P\{|X_n - X| \ge \varepsilon\}}_{\downarrow 0}$$

$X \in B_\varepsilon := \{y \in \mathbb{R} : \exists x \in B \text{ s.t. } |x-y| < \varepsilon\}$ (ε-neighborhood)

$$\Rightarrow \limsup_n P\{X_n \in B\} \le \underbrace{P\{X \in B_\varepsilon\}}_{\downarrow \text{(continuity of probability)}}.$$

- Let $\varepsilon \downarrow 0 \Rightarrow B_\varepsilon \downarrow \overline{B} \Rightarrow P\{X \in \overline{B}\}$

$$\Rightarrow \limsup_n P\{X_n \in B\} \le P\{X \in \overline{B}\} = P\{X \in B^\circ\} \le P\{X \in B\}.$$

$\overline{B} = B^\circ \sqcup \partial B$ (B is a continuity set)

- Apply this argument for B^c instead of B, which must also be a continuity set $(\partial(B^c) = \partial B) \Rightarrow \limsup_n P\{X_n \in B^c\} \le P\{X \in B^c\}$
$\Rightarrow \liminf_n P\{X_n \in B\} \ge P\{X \in B\}$. Combine upper & lower bds $\Rightarrow$
$P\{X_n \in B\} \longrightarrow P\{X \in B\}$. D.

(iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i) Since $P\{|X| > M\} \longrightarrow 0$ as $M \to \infty$ (continuity of measure), $\forall \varepsilon > 0$ $\exists M$ s.t. $P\{|X| > M\} < \varepsilon$ and $P\{|X| = M\} = 0$

$$\Rightarrow P\{X_n \le x\} = P\{\underbrace{X_n \le x, |X_n| \le M}_{\text{closed interval}}\} + P\{\underbrace{|X_n| > M}_{\text{complement of a closed interval}}\}$$

$\to P\{X\le x, |X|\le M\} + P\{|X|>M\}$ whenever $P\{X=x\}=0$ (assm.)

$\le P\{X\le x\}+\varepsilon.$

$\Rightarrow \limsup_n P\{X_n\le x\} \le P\{X\le x\}+\varepsilon.$

A reverse ineq. follows by the same argument with $X_n \leftrightarrow X$. □

(i) $\Rightarrow$ (iv) By Skorokhod's a.s. representation thm (p. 111), we can assume WLOG

that $\qquad X_n \to X$ everywhere

Continuity of $h \Rightarrow \qquad h(X_n)\to h(X)$ everywhere.

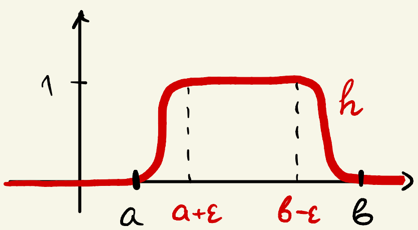
Boundedness of $h \xRightarrow{\text{DCT}} \qquad \mathbb{E}h(X_n)\to \mathbb{E}h(X).$ □

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (ii) Smoothing: Fix any $a<b$ satisfying $P\{X=a\}=P\{X=b\}=0$ (*)

Fix $\varepsilon>0$. $\exists$ function h as in the conclusion satisfying

$$\mathbb{1}_{[a+\varepsilon,\, b-\varepsilon]} \le h \le \mathbb{1}_{[a,b]} \text{ pointwise.}$$



Evaluate on X_n, take $\mathbb{E} \Rightarrow$

$$P\{X_n\in[a,b]\} \ge \mathbb{E}h(X_n)$$
$$\to \mathbb{E}h(X) \quad \text{(assumption)}$$
$$\ge P\{X\in[a+\varepsilon, b-\varepsilon]\}$$

$\Rightarrow \liminf_n P\{X_n\in[a,b]\} \ge P\{X\in[a+\varepsilon,b-\varepsilon]\}\ \forall \varepsilon>0$

Take $\varepsilon\downarrow 0 \Rightarrow$ by continuity of measure, RHS $\to P\{X\in(a,b)\} \overset{(*)}{=} P\{X\in[a,b]\}$.

$\Rightarrow \liminf_n P\{X_n\in[a,b]\} \ge P\{X\in[a,b]\}.$

Prove the upper bound similarly (DIY). □

SKIP HELLY, TIGHTNESS NEXT TIME?

Q: $\forall$ seq. of r.v's (X_n) has a weakly convergent subsequence ?

Ans: yes, almost. ~ "Bolzano-Weierstrass"

THM (Helly's selection thm) Let (F_n) be a sequence of nondecreasing, uniformly bounded functions $\mathbb{R} \to \mathbb{R}$. Then $\exists$ subsequence (F_{n_k}) and a nonincreasing, right-continuous function F such that

$$\lim_k F_{n_k}(x) = F(x) \quad \forall \text{ point of continuity } x \text{ of } F.$$

i.e. $\sup_n \|F_n\|_\infty < \infty$

Proof 1 (diagonal argument)

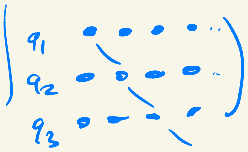
$\forall$ fixed x, Bolzano-Weierstrass $\Rightarrow \exists\, n_k = n_k(x)$:

$$(F_{n_k}(x)) \text{ converges.}$$

Diagonal argument $\Rightarrow \exists\, (n_k)$:

$$(F_{n_k}(q)) \text{ converges } \forall q \in \mathbb{Q}.$$

(Enumerate $\mathbb{Q} = \{q_1, q_2, \dots\}$. Find $(n_k^{(1)})$ giving convergence at q_1, $(n_k^{(2)}) \subset (n_k^{(1)})$ giving convergence on q_2, etc. Set $(n_k) := (n_k^{(k)})$.)



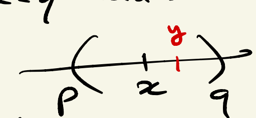
Let $\quad G(q) := \lim_k F_{n_k}(q) \qquad \forall q \in \mathbb{Q}$,

$\quad F(x) := \inf\{G(q) : q > x, q \in \mathbb{Q}\} \qquad \forall x \in \mathbb{R}$.

F is nonincreasing (by def).

2 (Sandwiching between 2 rationals)

Fix $\forall x \in \mathbb{R}$ and $\forall \varepsilon > 0$. $\exists\, p, q \in \mathbb{Q}$ such that $p < x < q$ and:



(*) $\quad F(x) > G(q) - \varepsilon \qquad$ (by def of F)

(**) $\quad F(x) < F(p) + \varepsilon \qquad$ if x is a pt of continuity of F.

3 F is right-continuous ? $\quad \forall y \in (x, q)$:

$$\begin{aligned} F(x) < F(y) &\quad \text{(monotonicity)} \\ \le G(q) &\quad \text{(def of } F\text{)} \\ < F(x) + \varepsilon &\quad \text{(by *)} \end{aligned}$$

$\Rightarrow |F(x) - F(y)| < \varepsilon \quad \square$

④ <u>If F is continuous at x, $F_{n_k}(x) \to F(x)$?</u>

$$
\begin{aligned}
F_{n_k}(x) &\le F_{n_k}(q) && \text{(monotonicity)} \\
&\to G(q) && \text{(def of } G) \\
&\le F(x)+\varepsilon && \text{(by } *) \qquad \Rightarrow \limsup_k F_{n_k}(x) \le F(x)+\varepsilon
\end{aligned}
$$

&

$$
\begin{aligned}
F_{n_k}(x) &\ge F_{n_k}(p) && \text{(monotonicity)} \\
&\to G(p) && \text{(def of } G) \\
&\ge F(p) && \text{(def of } F) \\
&> F(x)-\varepsilon && \text{(by } **). \qquad \Rightarrow \liminf_k F_{n_k}(x) \ge F(x)-\varepsilon. \qquad \varepsilon \downarrow 0 \Rightarrow \text{QED.}
\end{aligned}
$$

Ex 1. Show that convergence may <u>not</u> hold for <u>all</u> $x \in \mathbb{R}$ **HW**

2. Show that it does if we drop the right-continuity requirement for F

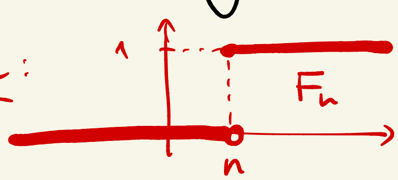
Q. <u>Apply Helly S.T for CDF's F_n</u>,

i.e. nondecreasing, right-continuous functions satisfying $F_n(-\infty)=0$, $F_n(+\infty)=1$.

Will the limit F be a CDF?

Ans: Not necessarily: $F(-\infty)=0$, $F(+\infty)=1$ may fail

<u>Example</u>:



$F_n \to 0$ pointwise, but 0 is NOT a CDF.

"MASS ESCAPES TO ∞"

Q: How can we prevent the escape?

Def (Tightness) A sequence of r.v's (X_n) is tight if $\forall \varepsilon>0 \ \exists M>0$:

$$P\{|X_n|>M\}<\varepsilon \quad \forall n.$$

Examples
- $X_n=n$ is not tight (see above)
- $X_n \sim \mathrm{Unif}[n,n+1]$ is not tight
- $X_n \sim N(0,n)$ is not tight (check! HW)
- $\forall$ r.v's with mean 0, variance 1 are tight (chebyshev)

Observation: If (F_n) in Helly's S.T. are CDF of tight (X_n) then $F(-\infty)=0$, $F(+\infty)=1$

Proof $\forall \varepsilon>0 \ \exists M>0$ (from def. of tightness)

$\forall x<-M$: $F(x)\le F(-M)$ (monotonicity)

$$= \lim_k F_{n_k}(-M) \quad \text{(Helly)}$$

$$= \lim_k \underbrace{P\{X_{n_k}\le -M\}}_{\le\varepsilon} \le \varepsilon \quad \Rightarrow F(-\infty)=0. \qquad \square$$

$\Downarrow$ [Prokhorov's theorem]

Cor ("Compactness") $\forall$ tight sequence of r.v's (X_n) $\exists$ subsequence (n_k) and a r.v. X such that

$$X_{n_k} \xrightarrow{w} X.$$