THE CENTRAL LIMIT THEOREM

We will prove,

THM (CLT) Let $X_1, X_2, \ldots$ be iid r.v's with mean μ and variance σ^2.

Then $S_n = X_1 + \cdots + X_n$ satisfies

$$\bar{S}_n := \frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{w} Z \sim N(0,1)$$

- By Portmanteau lemma (p.113), it is enough to show that

$$\mathbb{E}h(\bar{S}_n) \to \mathbb{E}h(Z)$$

$\forall$ "test function" $h: \mathbb{R} \to \mathbb{R}$ s.t. $\|h^{(k)}\|_\infty < \infty \quad \forall k = 0, 1, 2, \ldots$

Toward that goal:

Lem Let $X_1, X_2, \ldots$ be independent mean zero r.v's with $\mathbb{E}|X_k|^3 < \infty$.

Then $S_n := X_1 + \cdots + X_n$ satisfies

$$\left|\mathbb{E}h(S_n) - \mathbb{E}h(Z)\right| \le C\|h'''\|_\infty \sum_{k=1}^{n} \mathbb{E}|X_k|^3 \qquad \forall \text{ 3 times diff'ble } h,$$

where $Z \sim N(0, \mathrm{Var}(S_n))$ and C is an absolute constant.

Proof (Lyapunov-Lindeberg replacement method)

- $S_n = S_{n-1} + X_n$. WLOG $\|h'''\|_\infty = 1$. Taylor's approximation $\Rightarrow$

$$\left| h(S_n) - h(S_{n-1}) - \underbrace{X_n\, h'(S_{n-1})}_{\substack{\text{independent}\\ \Rightarrow\ \text{mean}=0}} - \underbrace{\frac{X_n^2}{2}\, h''(S_{n-1})}_{\text{independent}} \right| \le \frac{|X_n|^3}{3!}$$

Take $\mathbb{E}$ on both sides, use $|\mathbb{E} \cdots| \le \mathbb{E}|\cdots|$ (Jensen) $\Rightarrow$

$$\left|\mathbb{E}h(S_n) - \mathbb{E}h(S_{n-1}) - \frac{\sigma_n^2}{2}\mathbb{E}h''(S_{n-1})\right| \le \frac{\mathbb{E}|X_n|^3}{6} \qquad \text{where } \sigma_n^2 = \mathbb{E}X_n^2. \quad (*)$$

- Let Z_n be a r.v. with mean 0, $\mathbb{E}Z_n^2 = \sigma_n^2$, independent of $X_1, \ldots, X_n$

Arguing like in $(*)$ but replacing X_n by Z_n, we get

$$\left|\mathbb{E}h(S_{n-1} + Z_n) - \mathbb{E}h(S_{n-1}) - \frac{\sigma_n^2}{2}\mathbb{E}h''(S_{n-1})\right| \le \frac{\mathbb{E}|Z_n|^3}{6} \qquad (**)$$

Subtract (*) − (**) $\overset{\Delta}{\Rightarrow}$

$$\left|\mathbb{E}h(S_n) - \mathbb{E}h(S_{n-1}+Z_n)\right| \le \frac{1}{6}\Big(\mathbb{E}|\underbrace{X_n|^3 + \mathbb{E}|Z_n}_{\overset{\gtrsim}{\uparrow}}|^3\Big) \le C\cdot\mathbb{E}|X_n|^3.$$

$\Big(Z_n/\sigma_n \sim N(0,1) \Rightarrow \|Z_n/\sigma_n\|_{L^3} = 2\sqrt{\frac{2}{\pi}}$ (check!) $\Big) \Rightarrow \|Z_n\|_{L^3} \lesssim \sigma_n = \|X_n\|_{L^2} \le \|X\|_{L^3}\Big)$

- Continue the replacement process. $X_{n-1} \mapsto Z_{n-1} \Rightarrow$

$$\left|\mathbb{E}h(S_n) - \mathbb{E}h(S_{n-2}+Z_{n-1}+Z_n)\right| \le C\,\mathbb{E}|X_n|^3 + C\,\mathbb{E}|X_{n-1}|^3$$

$\cdots$ (n times) $\cdots \Rightarrow$

$$\left|\mathbb{E}h(S_n) - \mathbb{E}\Big(\sum_{k=1}^{n} Z_k\Big)\right| \le C\sum_{k=1}^{n}\mathbb{E}|X_k|^3$$

- $\sum_{k=1}^{n} Z_k$ is normal, mean $=0$, Variance $= \sum_{k=1}^{n}\sigma_k^2 = \mathrm{Var}(S_n)$ □

Proof of CLT assuming $\mathbb{E}|X_k|<\infty$:

WLOG $\mu=0$, $\sigma=1$, $\|h'''\|_\infty \le 1$.

Apply Lemma for $\frac{S_n}{\sqrt{n}} = \sum_{k=1}^{n}\frac{X_k}{\sqrt{n}}$ ← mean 0, variance 1 $\Rightarrow$

$$\left|\mathbb{E}h\Big(\frac{S_n}{\sqrt{n}}\Big) - \mathbb{E}h(\overset{N(0,1)}{Z})\right| \le C\sum_{k=1}^{n}\mathbb{E}\left|\frac{X_n}{\sqrt{n}}\right|^3 \overset{\text{iid}}{=} \frac{C\,\mathbb{E}|X_1|^3}{\sqrt{n}} \to 0 \text{ as } n\to\infty.$$

Portmanteau lemma (p.113) $\Rightarrow$ □.

Proof of CLT in full generality (Truncation):

- WLOG $\mu=0,\ \sigma=1,\ \|h'\|_\infty\le 1,\ \|h'''\|_\infty\le 1$. Fix $\forall\,\varepsilon>0$.

$Y_k := X_k\,\mathbb{1}_{\{|X_k|\le\varepsilon\sqrt{n}\}}$; $T_n := Y_1+\dots+Y_n$; $\mu_n := \mathbb{E}Y_k$; $\sigma_n^2 := Var(Y_k)$

DCT $\Rightarrow$ $\mu_n\to\mu=0$, $\sigma_n^2\to\sigma^2=1$ as $n\to\infty$.

- Apply Lem. p. 118 for $\frac{T_n-n\mu_n}{\sqrt{n}}=\sum_{i=1}^{n}\frac{Y_k-\mu_n}{\sqrt{n}}$ ← Mean 0, Variance σ_n^2 $\Rightarrow$

$$\left|\mathbb{E}h\left(\frac{T_n-n\mu_n}{\sqrt{n}}\right)-\mathbb{E}h(\overbrace{Z_n}^{N(0,\sigma_n^2)})\right|\le C\sum_{k=1}^{n}\mathbb{E}\left|\frac{Y_k-\mu_n}{\sqrt{n}}\right|^3=\frac{C\,\mathbb{E}|Y_1-\mathbb{E}Y_1|^3}{\sqrt{n}}$$

$$\le\frac{8C\,\mathbb{E}|Y_1|^3}{\sqrt{n}}\qquad\left(\|Y-\mathbb{E}Y\|_{L^3}\overset{\Delta}{\le}\|Y\|_{L^3}+\underbrace{|\mathbb{E}Y|}_{\le\,\mathbb{E}|Y|=\|Y\|_{L^1}\le\|Y\|_{L^3}}\le 2\|Y\|_{L^3}\right)$$

$$\le 8C\varepsilon.\qquad\left(\mathbb{E}|Y_1|^3\le\|Y_1\|_\infty\cdot\mathbb{E}Y_1^2\le\varepsilon\sqrt{n}\cdot\mathbb{E}X_1^2=\varepsilon\sqrt{n}\right)$$

- Let's get rid of truncation. ① $\mathbb{E}h\left(\frac{T_n-n\mu_n}{\sqrt{n}}\right)\overset{?}{\approx}\mathbb{E}h\left(\frac{S_n}{\sqrt{n}}\right)$ ② $\mathbb{E}h(Z_n)\overset{?}{\approx}\mathbb{E}h(Z)$

$$\left|\mathbb{E}h\left(\frac{T_n-n\mu_n}{\sqrt{n}}\right)-\mathbb{E}h\left(\frac{S_n}{\sqrt{n}}\right)\right|\le\mathbb{E}\left|h\left(\frac{T_n-n\mu_n}{\sqrt{n}}\right)-h\left(\frac{S_n}{\sqrt{n}}\right)\right|$$

$$\le\mathbb{E}\left|\frac{T_n-S_n-n\mu_n}{\sqrt{n}}\right|\qquad(\text{by assumption, }\|h'\|_\infty\le 1)$$

$$\overset{\text{Jensen}}{\le}\sqrt{\mathbb{E}\left(\frac{\overset{\text{mean }n\mu_n}{T_n}-\overset{\text{mean }0}{S_n}-n\mu_n}{\sqrt{n}}\right)^2}=\sqrt{\frac{Var(T_n-S_n)}{n}}$$

$$=\sqrt{Var\left(X_1\,\mathbb{1}_{\{|X_1|>\varepsilon\sqrt{n}\}}\right)}\qquad(\text{by def})$$

$$\le\sqrt{\mathbb{E}X_1^2\,\mathbb{1}_{\{|X_1|>\varepsilon\sqrt{n}\}}}\longrightarrow 0\ \text{ as } n\to\infty \text{ by DCT.}$$

☺

② $\underset{N(0,\sigma_n^2)}{Z_n}\overset{dist}{=}\underset{1}{\sigma_n}\underset{N(0,1)}{Z}\xrightarrow{a.s.}Z\ \Rightarrow\ Z_n\xrightarrow{w}Z\ \Rightarrow\ \mathbb{E}h(Z_n)\to\mathbb{E}h(Z)$ ☺

- (*) & ① & ② $\overset{\Delta}{\Longrightarrow}\ \limsup_n\left|\mathbb{E}h\left(\frac{S_n}{\sqrt{n}}\right)-\mathbb{E}h(Z)\right|\le\varepsilon$ ← arbitrary $\Rightarrow$ □.

Remarks on Lem p.118 :

- It is a nonasymptotic version of CLT.
- $\|f'''\|_\infty$ can be replaced by $\|f'\|_\infty$ (later - Stein's method)
- Third moment can be replaced by $(2+\varepsilon)$-moment.
  But the bound gets weaker [Khoshnevisan, Probability. Problem 7.44]

Remarks on the replacement method :

- It works in wider generality than sums of r.v's
  e.g. in RMT, see e.g. [Chatterjee], [Van-Vu 4th moment thm]