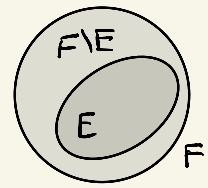
ELEMENTARY PROPERTIES OF PROB. MEASURES

Prop. 1 (Monotonicity) $E \subset F \Rightarrow \mathbb{P}(E) \leq \mathbb{P}(F)$



Proof 1: sets

$$F = E \sqcup (F \setminus E) \overset{\text{additivity}}{\Longrightarrow} \mathbb{P}(F) = \mathbb{P}(E) + \underbrace{\mathbb{P}(F\setminus E)}_{\overset{\text{(def)}}{\geq 0}} \geq \mathbb{P}(E). \quad \text{QED.}$$

↑ disjoint union

Proof 2: integrals $\quad \mathbb{1}_E(\omega) := \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{if not} \end{cases}$, and similarly for $\mathbb{1}_F$

$$E \subset F \Rightarrow \mathbb{1}_E(\omega) \leq \mathbb{1}_F(\omega) \quad \forall \omega \in \Omega$$

$$\Rightarrow \int \mathbb{1}_E \, d\mathbb{P} \leq \int \mathbb{1}_F \, d\mathbb{P}$$

(def of Lebesgue integral) $\rightarrow$ $\int \mathbb{1}_E \, d\mathbb{P} = \mathbb{P}(E)$, $\int \mathbb{1}_F \, d\mathbb{P} = \mathbb{P}(F)$ $\qquad$ QED

This is a circular argument. Monotonicity of integral may have been proved using monotonicity of measures.

Def (set limits) Events $E_i \uparrow E$ if $E_1 \subset E_2 \subset \dots$ and $\bigcup_{i=1}^{\infty} E_i = E$

$E_i \downarrow E$ if $E_1 \supset E_2 \supset \dots$ and $\bigcap_{i=1}^{\infty} E_i = E$

Prop 2 (Continuity) $\quad E_i \uparrow E \Rightarrow \mathbb{P}(E_i) \uparrow \mathbb{P}(E)$

$E_i \downarrow E \Rightarrow \mathbb{P}(E_i) \downarrow \mathbb{P}(E)$

Proof 1 (integrals): $\quad E_i \uparrow E \Rightarrow \mathbb{1}_{E_i}(\omega) \uparrow \mathbb{1}_E(\omega) \quad \forall \omega \in \Omega \quad$ (check!)

Monotone Convergence Thm $\Rightarrow \int \mathbb{1}_{E_i} \, d\mathbb{P} \uparrow \int \mathbb{1}_E \, d\mathbb{P}$, where $\int \mathbb{1}_{E_i} \, d\mathbb{P} = \mathbb{P}(E_i)$ and $\int \mathbb{1}_E \, d\mathbb{P} = \mathbb{P}(E)$. $\qquad$ QED

Proof 2 (sets) – **HW ?**

This is a circular argument! MCT may have been proven using continuity of measures

Prop. 2 (Union Bound) $\quad \mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(E_i)$

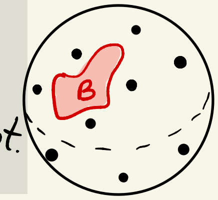
Proof 1 (integrals): $\quad E := \bigcup_{i=1}^{\infty} E_i \Rightarrow \mathbb{1}_E(\omega) \leq \sum_{i=1}^{\infty} \mathbb{1}_{E_i}(\omega) \quad$ (check!)

Monotone Conv. Thm $\Rightarrow \int \mathbb{1}_E \, d\mathbb{P} \leq \sum_{i=1}^{\infty} \underbrace{\int \mathbb{1}_{E_i} \, d\mathbb{P}}_{=\mathbb{P}(E_i)}$, where $\int \mathbb{1}_E \, d\mathbb{P} = \mathbb{P}(E)$ $\qquad$ QED

Proof 2 (sets) **HW** (express E as a disjoint union of $F_1 = E_1$, $F_2 = E_2 \setminus F_1$, $F_3 = E_3 \setminus F_2, \dots$)

Example (Probabilistic Method)

Mark $\forall$ n dots on a soccer ball.
Let B be $\forall$ sticker whose area is $< \frac{1}{n}$ of the ball's surface area.
Then B can be placed on the ball so it does not cover any dot.



S^{d-1}

Proof Let the dots be $x_1, \dots, x_n \in S^{d-1}$.

Choose $U \in SO(n)$ at random, according to Haar measure

- $\forall i$: $\mathbb{P}\{\underbrace{x_i \in U(B)}_{\overset{\shortparallel}{E_i}}\} = \mathbb{P}\{\underbrace{U^{-1}(x)}_{\sim\ \mathrm{Unif}(S^{d-1})} \in B\} = \dfrac{\mathrm{Area}(B)}{\mathrm{Area}(S^{d-1})} < \dfrac{1}{n}$.

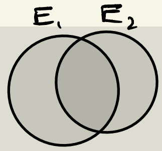
  [uniqueness of rotation-invariant prob. measure on S^{d-1}]

- Union bound:

$$\mathbb{P}\{\exists\, i \in \{1,\dots,n\} : x_i \in U(B)\} = \mathbb{P}\Big(\bigcup_{i=1}^{n} E_i\Big) \le \sum_{i=1}^{n} \mathbb{P}(E_i) < n \cdot \frac{1}{n} = 1.$$

$$\Rightarrow \mathbb{P}\{\forall i \in \{1,\dots,n\} : x_i \notin U(B)\} \qquad \Rightarrow \quad \exists \text{ such } U. \qquad \square$$

Lem (Inclusion-Exclusion Principle)



- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
- $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$
- $P\Big(\bigcup_{i=1}^{n} E_i\Big) = \underbrace{\sum_{i=1}^{n} P(E_i)}_{n \text{ terms}} - \underbrace{\sum_{i<j} P(E_i \cap E_j)}_{\binom{n}{2} \text{ terms}} + \underbrace{\sum_{i<j<k} P(E_i \cap E_j \cap E_k)}_{\binom{n}{3} \text{ terms}} - \dots + \underbrace{(-1)^{n-1} P(E_1 \cap \dots \cap E_n)}_{\binom{n}{n} = 1 \text{ term}}$

Proof for $n=2$: $E := E_1 \cup E_2 \Rightarrow (\mathbb{1}_E - \mathbb{1}_{E_1})(\mathbb{1}_E - \mathbb{1}_{E_2})(\omega) = 0 \quad \forall \omega \in \Omega$

$$\Rightarrow \quad (\mathbb{1}_{E \cap E} - \mathbb{1}_{E_1 \cap E} - \mathbb{1}_{E \cap E_2} + \mathbb{1}_{E_1 \cap E_2})(\omega) = 0$$

$$\Rightarrow \quad (\mathbb{1}_E - \mathbb{1}_{E_1} - \mathbb{1}_{E_2} + \mathbb{1}_{E_1 \cap E_2})(\omega) = 0$$

Rearrange terms $\Rightarrow$ $\mathbb{1}_E(\omega) = (\mathbb{1}_{E_1} + \mathbb{1}_{E_2} - \mathbb{1}_{E_1 \cap E_2})(\omega) \quad \forall \omega \in \Omega$

Integrate $\Rightarrow$ $\underbrace{\int \mathbb{1}_E \, dP}_{\overset{\shortparallel}{P(E)}} = \underbrace{\int \mathbb{1}_{E_1} \, dP}_{\overset{\shortparallel}{P(E_1)}} + \underbrace{\int \mathbb{1}_{E_2} \, dP}_{\overset{\shortparallel}{P(E_2)}} - \underbrace{\int \mathbb{1}_{E_1 \cap E_2} \, dP}_{\overset{\shortparallel}{P(E_1 \cap E_2)}}$

HW: prove for $\forall$ n

Q. (derangements) n exams are returned to n students at random. What is the probability that no student receives their own exam?

$\iff P(\text{a random permutation } \pi \in S_n \text{ has no fixed point}) = ?$

"derangement"

Sol: Complement: $E :=$ "at least one student gets own exam"

$$E = \bigcup_{i=1}^{n} E_i \ , \text{ where } E_i = \text{"student } i \text{ gets own exam"}$$

Use IEP:

(1) $P(E_i) = \frac{|E_i|}{|S|} = \frac{(n-1)!}{n!} = \frac{1}{n}$ ← # of ways to return n-1 exams (after student i gets their own)

(2) $P(E_i \cap E_j) = P\{\text{both students } i,j \text{ get their own exams}\}$

$$= \frac{|E_i \cap E_j|}{|S|} = \frac{(n-2)!}{n!}$$ ← # ways to return n-2 exams (after studs i,j get their own)

(3) $P(E_i \cap E_j \cap E_k) = \frac{(n-3)!}{n!}$

... IEP ⟹

$$P(E) = n \cdot \frac{1}{n} - \underbrace{\binom{n}{2}}_{=\frac{n!}{2!(n-2)!}} \frac{(n-2)!}{n!} + \underbrace{\binom{n}{3}}_{=\frac{n!}{3!(n-3)!}} \frac{(n-3)!}{n!} - \cdots + (-1)^{n-1} \binom{n}{n} \frac{(n-n)!}{n!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n-1} \frac{1}{n!}$$

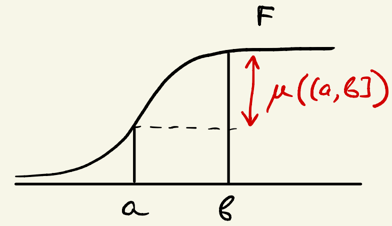
$$\Rightarrow P(E^c) = 1 - P(E) = \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \simeq \boxed{\frac{1}{e}} \approx 0.37.$$

(recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$)

Describing a probability measure with a distribution function

Recall the def of Lebesgue-Stieltjes measure :

THM $\forall$ nondecreasing, right-continuous function $F: \mathbb{R} \to \mathbb{R}$
$\exists!$ Borel measure μ on $\mathbb{R}$ s.t.
$$\mu\big((a,b]\big) = F(b) - F(a) \quad \forall\, a \le b$$



Ex $F(x) = x \Rightarrow$ Lebesgue measure.

Note : if $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$ then μ is a probability measure

$$\left.\begin{aligned} &(-n,n] \uparrow \mathbb{R} \Rightarrow \text{by continuity, } \mu\big((-n,n]\big) \uparrow \mu(\mathbb{R}) \\ &\qquad\qquad\qquad\qquad \mu\big((-n,n]\big) = F(n) - F(-n) \to 1 - 0 = 1 \end{aligned}\right\} \Rightarrow \mu(\mathbb{R}) = 1$$

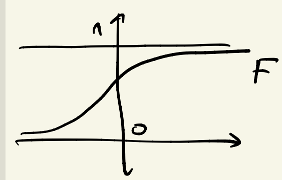
This justifies:

Def A distribution function is a function $F: \mathbb{R} \to [0,1]$ such that :

(i) F is nondecreasing;

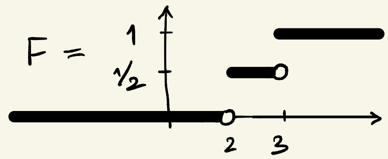
(ii) F is right-continuous;

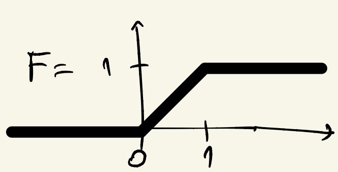
(iii) $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to+\infty} F(x) = 1$.



We proved :

THM $\forall$ distribution function F $\exists!$ Borel probability measure μ on $\mathbb{R}$ such that
$$\mu\big((a,b]\big) = F(b) - F(a) \qquad \forall\, a \le b.$$

Ex :(a) if $F =$  then $\mu(\{2\}) = \mu(\{3\}) = \frac{1}{2}$.

(b) if $F =$  then $\mu =$ Lebesgue on $[0,1]$.

Conversely:

THM $\forall$ Borel probability measure μ $\exists!$ distribution function F such that

$$\mu\big((a,b]\big) = F(b) - F(a) \qquad \forall\, a \le b.$$

Proof Define

$$F(x) := \mu\big((-\infty, x]\big), \qquad x \in \mathbb{R}$$

Then: (i) F is nondecreasing (monotonicity of measures)

(ii) F is right-continuous

$\forall\, y_n \downarrow x$, $(-\infty, y_n] \downarrow (-\infty, x]$

$\Rightarrow \mu\big((-\infty, y_n]\big) \downarrow \mu\big((-\infty, x]\big)$ (continuity of measures)

$\Rightarrow F(y_n) \downarrow F(x)$.

(iii) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$

Let's prove this.

If $x_n \to +\infty$, $(-\infty, x_n] \uparrow \mathbb{R}$

$\Rightarrow \mu\big((-\infty, x_n]\big) \uparrow \mu(\mathbb{R}) = 1$ (continuity of measures)

$\Rightarrow F(x_n) \uparrow 1$.

QED