Remarks on Lem p.118 :

- It is a nonasymptotic version of CLT.
- $\|f'''\|_\infty$ can be replaced by $\|f'\|_\infty$ (later - Stein's method)
- Third moment can be replaced by $(2+\varepsilon)$-moment. But the bound gets weaker (Khoshnevisan, Probability. Problem 7.44)

Remarks on the replacement method :

- It works in wider generality than sums of r.v's e.g. in RMT, see e.g. [Chatterjee], [Van-Vu 4^{th} moment thm]

Application for mean estimation: $\mu_n := \dfrac{X_1+\dots+X_n}{n} \overset{?}{\approx} \mu$. $\quad Var(\hat\mu) = \dfrac{\sigma}{\sqrt{n}}$

$$P\Big\{ |\underset{\underset{S_n/n}{\parallel}}{\mu_n} - \mu| \le \frac{t\sigma}{\sqrt{n}} \Big\} = P\Big\{ \Big|\frac{S_n - n\mu}{\sigma\sqrt{n}}\Big| \le t \Big\} \xrightarrow[CLT]{} P\{ |\underset{\underset{N(0,1)}{\wr}}{Z}| \le t \} = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-x^2} dx$$

Confidence interval.

FOURIER ANALYTIC PROOF OF CLT

Heuristic : • By Portmanteau lemma, $X_n \xrightarrow{w} X \iff \mathbb{E}h(X_n) \to \mathbb{E}h(X)$ $\forall$ "nice" h $(*)$

- It is enough to check $(*)$ for functions of the form $h(x) = e^{itx}$, i.e.

$$X_n \xrightarrow{w} X \iff \mathbb{E}e^{itX_n} \to \mathbb{E}e^{itX} \quad \forall t \in \mathbb{R} \qquad \text{(will prove.)}$$

- CLT for mean 0, var 1 :

$$\mathbb{E}e^{it(X_1+\dots+X_n)/\sqrt{n}} = \prod_{i=1}^{n} \mathbb{E}e^{itX_i/\sqrt{n}} = \left(\mathbb{E}e^{itX/\sqrt{n}}\right)^n \ominus$$

- Taylor: $\mathbb{E}e^{itx} \approx 1 + itx + \dfrac{(itx)^2}{2}$. Substitute $x = X$, take $\mathbb{E} \Rightarrow$

$$\mathbb{E}e^{itX} \approx 1 - \frac{t^2}{2}$$

$$\ominus \left(1 - \frac{t^2}{2n}\right)^n \to e^{-t^2/2} = \mathbb{E}e^{itZ} \quad \text{where } Z \sim N(0,1) \qquad \square.$$

- $\mathbb{E}e^{itX} = \int e^{itx} \underset{\underset{\text{pdf}}{\uparrow}}{f(x)} dx$ = Fourier transform of f.

Formally :

Fourier Analysis Background :

Def Formally, for $f : \mathbb{R} \to \mathbb{R}$

$$\hat{f}(t) := \int_{-\infty}^{\infty} e^{-i 2\pi t x} f(x)\,dx, \qquad t \in \mathbb{R}.$$

Facts (Check!)

① $f \in S \Rightarrow \hat{f} \in S$ where

S = Schwartz space $= \{ f : \mathbb{R} \to \mathbb{R} : \| x^p f^{(q)}(x) \|_\infty < \infty \ \forall p, q = 0, 1, 2, \dots \}$

② $f \in S$ is uniquely determined by $\hat{f} \in S$:

$$f(x) = \int_{-\infty}^{\infty} e^{i 2\pi t x} \hat{f}(t)\,dt, \qquad x \in \mathbb{R}$$

"Fourier inversion formula"

- In probabilistic terms :

Def The characteristic function of a r.v. X is

$$\phi_X(t) = \phi(t) = \mathbb{E} e^{itX}, \qquad t \in \mathbb{R}$$

- Connection to Fourier transform :

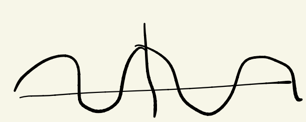
if X has density $f(x)$, $\quad \phi(2\pi t) = \int_{-\infty}^{\infty} e^{i 2\pi t x} f(x)\,dx = \hat{f}(t)$

Generally, $\phi(2\pi t) = \int_{-\infty}^{\infty} e^{i 2\pi t x}\, d\mathcal{L}_X(x)$ defines Fourier transform of a measure.

Fact If X, Y are independent, $\quad \phi_{X+Y} = \phi_X \cdot \phi_Y$

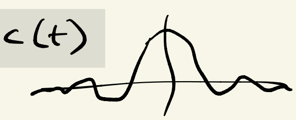
$\ulcorner e^{it(X+Y)} = e^{itX} \cdot e^{itY}$ take $\mathbb{E}$. $\lrcorner$

Examples (a) $X \sim$ Rademacher $\Rightarrow \phi(t) = \dfrac{e^{it} + e^{-it}}{2} = \boxed{\cos t}$



(b) $X \sim \mathrm{Unif}[-1,1] \Rightarrow \phi(t) = \int_{-1}^{1} e^{itx} \cdot \frac{1}{2}\,dx = \dfrac{\sin t}{t} = \boxed{\mathrm{sinc}(t)}$

change var $y = itx$



(c) $X \sim N(0,1) \Rightarrow \phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{itx} e^{-x^2/2}\,dx = \boxed{e^{-t^2/2}}$ (check!)

- Characteristic function ϕ_X determines the distribution of X uniquely. Moreover, stability holds:

THM (Lévy's continuity theorem)

$$X_n \xrightarrow{w} X \iff \phi_{X_n} \to \phi_X \text{ pointwise.}$$

$\Rightarrow$ In particular, uniqueness:

$X \overset{dist}{=} Y \iff \phi_X = \phi_Y$ everywhere.

Proof

($\Rightarrow$) $\forall$ fixed t, $x \mapsto e^{itx}$ is bdd, continuous.

$\Rightarrow$ by Portmanteau lemma, $\mathbb{E}e^{itX_n} \to \mathbb{E}e^{itX}$. $\square$

($\Leftarrow$) By P.L., it is enough to show that

Schwartz space

$$\mathbb{E}h(X_n) \to \mathbb{E}h(X) \quad \forall h \in S.$$

$$\big\| \quad \mathbb{E}\int_{-\infty}^{\infty} e^{i2\pi t X_n}\hat{h}(t)\,dt$$ (Fourier inversion formula p.123)

$$= \int_{-\infty}^{\infty} \mathbb{E}[e^{i2\pi t X_n}]\,\hat{h}(t)\,dt$$ (Fubini-Tonelli: $\hat{h} \in S$ by Fact 1 p.123, $|e^{i\cdots}| = 1$ $\Rightarrow$ absolutely integrable)

$$= \int_{-\infty}^{\infty} \phi_{X_n}(2\pi t)\,\hat{h}(t)\,dt$$

$$\xrightarrow{DCT} \int_{-\infty}^{\infty} \phi_X(2\pi t)\,h(t)\,dt$$ (by assumption; $|\phi_X(\cdot)| \le 1 \Rightarrow$ dominated by $h \in S$)

$$= \mathbb{E}h(X).$$ $\square$

Quick application of uniqueness:

① Sum of normals = normal: if $X_i \sim N(0, \sigma_i^2)$ are indep then $\sum_i X_i \sim N(0, \sigma^2)$, $\sigma^2 = \sum_i \sigma_i^2$

$\ulcorner \phi_{X_i}(t) = e^{-t^2\sigma_i^2/2} \Rightarrow \phi_{\sum X_i}(t) = \prod_i \phi_i(t) = \prod_i e^{-t^2\sigma_i^2/2} = e^{-t^2\sigma^2/2} = \phi_Z(t)$ for $Z \sim N(0,\sigma^2)$ $\lrcorner$

② Sum of Cauchy is Cauchy: If $X_i \sim$ Cauchy are indep then $\frac{1}{n}\sum_i X_i \sim$ Cauchy

$\ulcorner \phi_{X_i}(t) = e^{-|t|}$ (check!) $\Rightarrow$ repeat the argument in ①. $\lrcorner$

THM (Taylor expansion of the ch. function)

Let X be a r.v. with mean μ, variance σ^2. Then
$$\phi_X(t) = 1 + it\mu - \frac{t^2\sigma^2}{2} + o(t^2) \quad \text{as } t \to 0.$$

Proof • Taylor expansion:
$$e^{ix} = 1 + ix - \frac{(ix)^2}{2!} + \cdots$$

More precisely:
$$\boxed{\left| e^{ix} - 1 - ix + \frac{x^2}{2} \right| \le 4 \min(|x|^3, x^2) \quad \forall x \in \mathbb{R}}$$

Taylor with Lagrange remainder: since all derivatives of e^{ix} are $|\cdot| = 1$

$$\Rightarrow \left| e^{ix} - 1 - ix + \frac{x^2}{2} \right| \le \frac{|x|^3}{3!}$$

If $|x| \le 1$, RHS $\le \min(|x|^3, x^2)$

If $|x| \ge 1$, LHS $\overset{\triangle}{\le} |e^{ix}| + 1 + |ix| + \frac{x^2}{2} = 2 + x + \frac{x^2}{2} \le 4x^2$

• Substitute $x = tX$, take $\mathbb{E}$ $\Rightarrow$

$$\left| \phi_X(t) - 1 - it\mu + \frac{t^2\sigma^2}{2} \right| \overset{\text{Jensen}}{\le} \mathbb{E} \left| e^{itX} - 1 - itX + \frac{t^2X^2}{2} \right| \le 4\, \mathbb{E} \min(t^3|X|^3, t^2|X|^2)$$
$$= 4t^2 \cdot \mathbb{E} \underbrace{\min(t|X|^3, X^2)}$$

this r.v. is $\le X^2$ and $\to 0$ everywhere as $t \to 0$

DCT $\Rightarrow$ $\mathbb{E} \min(\cdot) \to 0$ as $t \to 0$. $\square$

Examples: (a) $X \sim N(0, \sigma^2) \Rightarrow \phi_X(t) = e^{-t^2\sigma^2/2} \approx 1 - \frac{t^2\sigma^2}{2}$.

(b) $X \sim$ Cauchy $\quad \phi_X(t) = e^{-|t|}$. The cusp at 0 reflects $\nexists$ mean.

Let's reprove

Thm (CLT) Let $X_1, X_2, \dots$ be iid. r.v's with mean μ and variance σ^2.

Then $S_n = X_1 + \dots + X_n$ satisfies

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{w} Z \sim N(0,1)$$

Proof WLOG $\mu = 0$, $\sigma = 1$ $\Rightarrow$ By Thm p. 125,

$$\phi_{X_i}(\tau) = 1 - \frac{\tau^2}{2} + \underbrace{\varepsilon(\tau)}_{\to 0 \text{ as } \tau \to 0}\tau^2$$

Fix $\forall t \in \mathbb{R}$.

$$\phi_{S_n/\sqrt{n}}(t) = \phi_{S_n}(t/\sqrt{n}) = \prod_{i=1}^{n} \phi_{X_i}(t/\sqrt{n}) = \left(\phi_{X_1}(t/\sqrt{n})\right)^n$$

$$= \left[1 - \frac{t^2}{2n} + \varepsilon\left(\frac{t}{\sqrt{n}}\right) \cdot \frac{t^2}{n}\right]^n \quad \text{as } n \to \infty$$

$$= \left[1 - \underbrace{\frac{t^2}{2n}\left(1 - 2\varepsilon(\underbrace{t/\sqrt{n}}_{\to 0})\right)}_{\to 1}\right]^n \longrightarrow e^{-t^2/2}$$

Use Fact:

$z_n \to z$ in $\mathbb{C}$ $\Rightarrow \left(1 + \frac{z_n}{n}\right)^n \to e^z$

$$= \phi_Z(t) \quad \text{where } Z \sim N(0,1)$$

Lévy continuity thm $\Rightarrow$ $\frac{S_n}{\sqrt{n}} \xrightarrow{w} Z$. $\square$