CLT for non-iid r.v's

Lindeberg's CLT. For each $n = 1, 2, \ldots$ let $X_{n1}, X_{n2}, \ldots, X_{nn}$ be independent, mean zero r.v's. Assume that, as $n \to \infty$,

(i) $\displaystyle\sum_{k=1}^{n} \mathbb{E} X_{nk}^2 \to \sigma^2 \in (0, \infty)$

(ii) $\displaystyle\sum_{k=1}^{n} \mathbb{E} X_{nk}^2 \mathbb{1}_{\{|X_{nk}| > \varepsilon\}} \to 0 \quad \forall \varepsilon > 0$

Then $\displaystyle\underbrace{\sum_{k=1}^{n} X_{nk}}_{= S_n} \xrightarrow{w} Z \sim N(0, \sigma^2)$

Remarks

① Lind CLT $\Rightarrow$ classical CLT $\left(\text{take } X_{nk} = \frac{X_k}{\sqrt{n}}\right.$; $\left.(ii) = \mathbb{E} X_1 \mathbb{1}_{\{|X_1| > \varepsilon\sqrt{n}\}} \to 0 \text{ by DCT}\right)$

② The variances $\sigma_{nk}^2 := \mathbb{E} X_{nk}^2$ satisfy $\boxed{\max_k \sigma_{nk}^2 \to 0}$ as $n \to \infty$

$$\mathbb{E} X_{nk}^2 = \underbrace{\mathbb{E} X_{nk}^2 \mathbb{1}_{\{|X_{nk}| \le \varepsilon\}}}_{\le \varepsilon} + \mathbb{E} X_{nk}^2 \mathbb{1}_{\{|X_{nk}| > \varepsilon\}}$$

$\Rightarrow \max_k \mathbb{E} X_{nk}^2 \le \varepsilon^2 + o(1)$ as $n \to \infty$ by (ii). ε is arbitrary $\Rightarrow$ ☐.

Proof same idea as in Classical CLT.

Levy's Continuity Thm (p.124) $\Rightarrow$ enough to prove

$$\varphi_{S_n}(t) \xrightarrow{?} \varphi_Z(t) = e^{-t^2\sigma^2/2} \quad \forall t \in \mathbb{R}$$

$$\varphi_{S_n}(t) = \mathbb{E} e^{itS_n} = \prod_{k=1}^{n} e^{itX_{nk}}$$

- To approximate each φ_{nk}, use Taylor's approx. of e^{itx} (p.126):

$$\left|e^{itx}-\left(1+ix-\frac{x^2}{2}\right)\right| \le 4\min(\underbrace{|x|^3}_{\overset{\wedge\!\!\!|}{4ax^2 \text{ if } |x|\le a}}, x^2) \quad \forall x\in\mathbb{R}.$$

$$\le 4ax^2 + 4x^2 \mathbb{1}_{\{|x|\ge a\}} \quad \forall a>0$$

Let X be a r.v. with mean 0, variance σ^2

Substitute $x=tX$, take $\mathbb{E}$, use Jensen. In RHS, choose $a=t\varepsilon$:

$$\left|\mathbb{E}e^{itX}-\underbrace{\left(1-\frac{t^2\sigma^2}{2}\right)}_{\substack{\text{SS} \\ \exp\left(-\frac{t^2\sigma^2}{2}\right)\pm\left(\frac{t^2\sigma^2}{2}\right)^2}}\right| \le 4t^3\varepsilon\sigma^2 + 4t^2\mathbb{E}\left[X^2\mathbb{1}_{\{|X|>\varepsilon\}}\right]$$

(use $|e^{-x}-(1-x)|\le x^2 \ \forall x\ge 0$)

Use for $X=X_{nk}$: mean 0, variance $=: \sigma_{nk}$ $\Rightarrow$

$$\Big|\underbrace{\mathbb{E}e^{itX_{nk}}}_{\substack{\shortparallel \\ a_k}}-\underbrace{\exp\left(-\frac{t^2\sigma_{nk}^2}{2}\right)}_{\substack{\shortparallel \\ b_k}}\Big| \le 4t^3\varepsilon\sigma_{nk}^2 + 4t^2\mathbb{E}\left[X_{nk}^2\mathbb{1}_{\{|X_{nk}|>\varepsilon\}}\right]+t^4\sigma_{nk}^4$$

$$\Big|\underbrace{\prod_{k=1}^n a_k}_{\varphi_{S_n}(t)} - \underbrace{\prod_{k=1}^n b_k}_{\exp\left(-\frac{t^2}{2}\underbrace{\sum_{k=1}^n\sigma_{nk}^2}_{\to\sigma^2}\right)}\Big| \le \sum_{k=1}^n |a_k-b_k|$$

(This ineq. holds $\forall$ complex numbers with moduli ≤ 1. [DIY])

$$\le 4t^3\varepsilon\underbrace{\sum_{k=1}^n\sigma_{nk}^2}_{\to\sigma^2}+4t^2\underbrace{\sum_{k=1}^n\mathbb{E}\left[X_{nk}^2\mathbb{1}_{\{|X_{nk}|>\varepsilon\}}\right]}_{\to 0 \text{ (assumption ii)}}+t^4\underbrace{\max_k(\sigma_{nk}^2)}_{\to 0 \text{ (Rem 2 p.128)}}\cdot\overbrace{\sum_{k=1}^n\sigma_{nk}^2}^{\to\sigma}$$

$$\limsup_{n\to\infty}\left|\varphi_{S_n}(t)-\exp\left(-\frac{t^2\sigma^2}{2}\right)\right|\le 4t^3\varepsilon\sigma^2 \quad \forall\varepsilon>0$$

$$\Rightarrow \ =0. \qquad \text{QED.}$$

Example (De Moivre-Laplace CLT)

$Y_{nk} \sim Ber(p_n)$ indep. $\Rightarrow T_n = \sum_{k=1}^{n} Y_{nk} \sim Binom(n, p_n)$

Assume $Var(T_n) = np_n(1-p_n) \to \infty$ as $n \to \infty$

Apply Lind CLT for $X_{nk} := \frac{Y_{nk} - p_n}{\sqrt{np_n(1-p_n)}}$

$$\begin{cases} \text{(i)} \quad \sum_{k=1}^{n} EX_{nk}^2 = 1 \quad \checkmark \\ \text{(ii)} \quad |X_{n1}| \leq \frac{1}{\sqrt{Var(T_n)}} \to 0 \quad \Rightarrow \text{the sum in (ii)} = 0 \text{ for large } n. \quad \checkmark \end{cases}$$

$\Rightarrow \quad S_n = \frac{T_n - np_n}{\sqrt{np_n(1-p_n)}} \xrightarrow{w} Z \sim N(0,1) \Rightarrow$ renaming T_n to S_n:

THM Let $S_n \sim Binom(n, p_n)$. If $Var(S_n) \to \infty$ then

$$\frac{S_n - ES_n}{\sqrt{Var(T_n)}} \xrightarrow{w} N(0,1)$$

What happens if $Var(S_n) \longrightarrow$ const ?

$\downarrow$

Poisson Limit Theorem

- let $X_{nk} \sim Ber(p_n)$, $\quad S_n = \sum_{k=1}^{n} X_{nk} \sim Binom(n, p_n)$

  Assume $p_n \le 1/2$ (by symmetry) and $\boxed{np_n \to \lambda = \text{const}}$ as $n \to \infty$

  i.e. const # of expected successes

- $\forall$ fixed $k = 0, 1, \dots, n$:

  needs justification

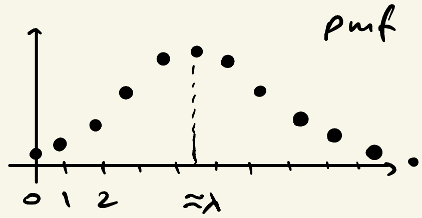
$$P\{S_n = k\} = \binom{n}{k} p_n^k (1-p_n)^{n-k} \approx \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

1 ($n \to \infty$, k fixed)

$$= \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{\lambda^k}{n^k} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \left(1 - \frac{\lambda}{n}\right)^{-k} \to e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Def (Poisson distribution) $X \sim Poisson(\lambda)$ if

$$P\{X = k\} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$



$\mathbb{E}X = \lambda$, $Var(X) = \lambda$ (DIY)

- We showed that pmf$(S_n) \to$ pmf(Poisson) pointwise.

  Scheffé Theorem (HW) $\Rightarrow$ $S_n \xrightarrow{w}$ Poisson.

• Alternative, Fourier-theoretic proof:

Poisson Limit Theorem Let $S_n \sim \mathrm{Binom}(n, p_n)$, $p_n \le 1/2$.
If $np_n \to \lambda$ as $n \to \infty$, then
$$S_n \xrightarrow{w} X \sim \mathrm{Poisson}(\lambda).$$

Proof By Levy Continuity Thm, enough to show that
$$\varphi_{S_n} \longrightarrow \varphi_X \quad \text{pointwise.}$$

① $$\varphi_X(t) = \mathbb{E}e^{itX} = \sum_{k=0}^{\infty} e^{itk} \cdot e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!}$$
$$= e^{-\lambda} \cdot \exp(\lambda e^{it}) = \boxed{\exp[\lambda(e^{it}-1)]}$$

② $$\varphi_{S_n}(t) = \mathbb{E}e^{it(\underbrace{X_{n1} + \dots + X_{nn}}_{\mathrm{Ber}(p_n)\text{ iid}})} = \left(\mathbb{E}e^{itX_{n1}}\right)^n$$
$$= \left[e^{it\cdot 1} p_n + e^{it\cdot 0}(1-p_n)\right]^n = \left[1 + \underbrace{p_n}_{\to \lambda/n}(e^{it}-1)\right]^n$$
$$\to \exp\left[\lambda(e^{it}-1)\right] = \varphi_X(t). \qquad \square.$$

SUMMARY:

THM (Gaussian-Poisson dichotomy)
Let $S_n \sim \mathrm{Binom}(n, p_n)$ with $p_n \le 1/2$.

(i) If $\mathbb{E}S_n = np_n \to \infty$ then $\dfrac{S_n - np_n}{\sqrt{np_n(1-p_n)}} \xrightarrow{w} N(0,1)$ ← rare successes

(ii) If $\mathbb{E}S_n \to \lambda = \text{const}$ then $S_n \to \mathrm{Poisson}(\lambda)$ ← frequent successes.

HW: Poisson $(\lambda) \to$ normal if $\lambda \to \infty$