MARTINGALES

Let $(\Omega, \Sigma, \mathbb{P})$ be a prob. space.

Def Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subset \Sigma$ be a sequence of σ-algebras (a "filtration")
A sequence of integrable r.v's $(X_1, X_2, \dots)$ is called a martingale if $\forall n \in \mathbb{N}$:

(i) X_n is $\mathcal{F}_n$-measurable ("(X_n) is adapted to $(\mathcal{F}_n)$")

(ii) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ a.s.

- Interpretation: n = time $\Rightarrow$ (X_n) is a stochastic process.
  $\mathcal{F}_n$ = information available at time n

EXAMPLES:

(a) Simple random walk: $S_n = Y_1 + \dots + Y_n$, $Y_i \sim$ Rademacher iid

is a martingale w.r.to $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$

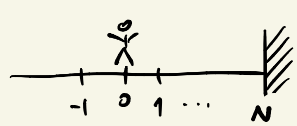
$$\mathbb{E}[\underbrace{S_{n+1}}_{= S_n + Y_{n+1}} \mid Y_1, \dots, Y_n] = \mathbb{E}[\underbrace{S_n}_{\text{mble}} \mid Y_1, \dots, Y_n] + \mathbb{E}[Y_{n+1} \mid \underbrace{Y_1, \dots, Y_n}_{\text{independent}}]$$

$$= S_n + \mathbb{E}[Y_{n+1}] = S_n.$$

(b) More generally, partial sums of $\forall$ mean zero r.v's

(c) Random walk with a sticky wall:

$$X_0 := 0; \qquad X_{n+1} := \begin{cases} X_n + \underbrace{Y_{n+1}}_{\text{iid Rademachers}} & \text{if } X_n < N \\ X_n & \text{if } X_n = N \end{cases}$$



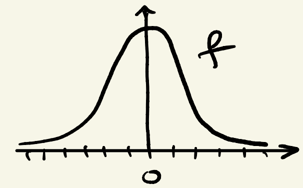
(i) X_n is $\sigma(Y_1, \dots, Y_n)$-mble (X_n is determined by $Y_1, \dots, Y_n$)

(ii) $\mathbb{E}[X_{n+1} \mid Y_1, \dots, Y_n] = X_n$ in either case.

- Convenient to express as $\boxed{X_n = S_{n \wedge T}}$
  where $S_n = Y_1 + \dots + Y_n$ is a simple random walk,
  $T = \min\{k : S_k = N\}$ is a stopping time

(d) More generally, a <u>slowed-down random walk</u>:

$$X_{n+1} := X_n + Y_{n+1} f(X_n) \qquad \forall f : \mathbb{R} \to \mathbb{R}, \text{ e.g.}$$



(e) <u>Quadratic martingale</u>:

> If $S_n = Y_1 + \dots + Y_n$ where Y_i are iid mean 0 variance 1
> then $S_n^2 - n$ is a martingale.

$$\Big\lceil\ \mathbb{E}[\underbrace{S_{n+1}^2}_{=(S_n+Y_{n+1})^2} - (n+1) \mid Y_1, \dots, Y_n] = \mathbb{E}[\underbrace{S_n^2}_{\text{mble}} + 2\underbrace{S_n}_{\text{mble}} Y_{n+1} + \underbrace{Y_{n+1}^2}_{\text{indep}} \underbrace{- n - 1} \mid Y_1, \dots, Y_n]$$

$$= S_n^2 + 2 S_n \underbrace{\mathbb{E}[Y_{n+1} \mid Y_1, \dots, Y_n]}_{= 0 \text{ (indep.)}} + \underbrace{\mathbb{E}(Y_{n+1}^2]}_{=1} - n - 1 = S_n^2 - n. \ \Big\rfloor$$

(f) <u>Product martingale</u>

> If $Y_1, Y_2, \dots \geq 0$ be indep. r.v's with mean 1,
> then $X_n := Y_1 \cdots Y_n$ is a martingale.

$$\Big\lceil\ \mathbb{E}[\underbrace{X_{n+1}}_{= \underbrace{X_n}_{\text{mble}} Y_{n+1}} \mid Y_1, \dots, Y_n] = X_n\, \mathbb{E}[Y_{n+1} \mid Y_1, \dots, Y_n] = X_n \underbrace{\mathbb{E}[Y_{n+1}]}_{=1} = X_n \ \Big\rfloor$$

$\uparrow$ indep $\nearrow$

(g) in particular, <u>St. Petersburg martingale</u>:

$$X_0 := 1, \quad X_{n+1} \mid X_n = \begin{Bmatrix} 2X_n, & \text{prob. } 1/2 \\ 0, & \text{prob. } 1/2 \end{Bmatrix} \qquad \text{(double the bet)}$$

$$\Rightarrow X_n = Y_1 \cdots Y_n \quad \text{where} \quad Y_k = \begin{Bmatrix} 2, & \text{prob. } 1/2 \\ 0, & \text{prob. } 1/2 \end{Bmatrix} \text{ iid.} \quad \mathbb{E}[Y_k] = 1.$$

(h) de Moivre's martingale

If $S_n := Y_1 + \dots + Y_n$ is a nonsymmetric r.walk: $Y_k = \begin{cases} 1, & \text{prob. } p \\ -1, & \text{prob. } q = 1-p \end{cases}$

then $X_n := \left(\frac{q}{p}\right)^{S_n}$ is a martingale

Γ $X_n = \left(\frac{q}{p}\right)^{Y_1} \cdots \left(\frac{q}{p}\right)^{Y_n}$, $\mathbb{E}\left(\frac{q}{p}\right)^{Y_k} = \left(\frac{q}{p}\right)^{1} \cdot p + \left(\frac{q}{p}\right)^{-1} \cdot q = q + p = 1$

$\Rightarrow$ product martingale. $\lrcorner$

(i) likelihood ratio test

see [Walsh]

Assume a r.v. X has density f or g.
Decide which one, based on an iid sample $X_1, \dots, X_n \sim X$.

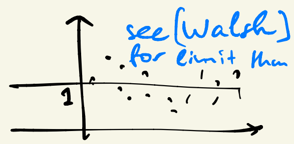
Solution: compute the likelihood ratio $\Lambda_n := \dfrac{\prod_{i=1}^n f(X_i)}{\prod_{i=1}^n g(X_i)}$

- If X has density g, then (Λ_n) is a martingale

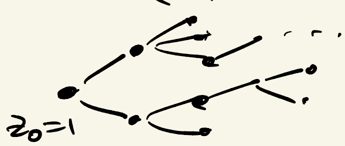
Γ $\Lambda_n = Y_1 \cdots Y_n$ where $Y_i = \dfrac{f(X_i)}{g(X_i)}$ are iid r.v's

with $\mathbb{E} Y_i = \int_{-\infty}^{\infty} \frac{f(x)}{\cancel{g(x)}} \cdot \cancel{g(x)}\, dx = 1$ $\Rightarrow$ product martingale. $\lrcorner$

- Hence, under the g-hypothesis, (Λ_n) does not "drift"



(j) Branching processes



Galton-Watson process: model of the size of population

Assume each member of n'th generation gives birth independently to a random # of children, μ on average.

$Z_n :=$ # individuals in n'th generation.

$Z_0 = 1$; $\mathbb{E}[Z_{n+1} | Z_n] = \mu Z_n$

$\Rightarrow \dfrac{Z_n}{\mu^n}$ is a martingale (divide by μ^n to check)

(k) Polya's urn

An urn initially contains N white and M black balls.
One ball is randomly drawn from an urn; its color is observed; it is returned to the urn, and an additional ball of the same color is added to the urn. Repeat.

$X_n :=$ fraction of white balls after n'th step is a martingale

If after n'th step the urn contains w white and b black balls,

$$X_{n+1} = \begin{cases} \dfrac{w+1}{w+b+1}, & \text{prob} = \dfrac{w}{w+b} \quad \leftarrow \text{white ball drawn} \\[2ex] \dfrac{w}{w+b+1}, & \text{prob} = \dfrac{b}{w+b} \quad \leftarrow \text{black ball drawn} \end{cases}$$

$$\mathbb{E}[X_{n+1} \mid w, b] = \frac{w+1}{w+b+1} \cdot \frac{w}{w+b} + \frac{w}{w+b+1} \cdot \frac{b}{w+b} = \frac{\cancel{(w+1+b)}\, w}{\cancel{(w+b+1)}(w+b)} = \frac{w}{w+b} = X_n$$

(l) Doob's martingale

Let X be a r.v. with $\mathbb{E}|X| < \infty$ and $(\mathcal{F}_n)$ be a filtration. Then

$$X_n := \mathbb{E}[X \mid \mathcal{F}_n]$$

is a martingale.

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\big[\, \mathbb{E}[X \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n \big] = \mathbb{E}[X \mid \mathcal{F}_n] = X_n$$

SIMPLE PROPERTIES

Prop. 1 $\mathbb{E}[X_n \mid \mathcal{F}_m] = \underset{\substack{\uparrow \\ \text{min}}}{X_{n \wedge m}} \quad \forall\, n, m \in \mathbb{N}$

$\ulcorner$ If $n \le m$ then X_n is $\mathcal{F}_n \subset \mathcal{F}_m$ -mble $\Rightarrow$ $\mathbb{E}[X_n \mid \mathcal{F}_m] = X_n$

If $n > m$ then $\mathbb{E}[X_n \mid \mathcal{F}_m] = \mathbb{E}\big[\underbrace{\mathbb{E}[X_n \mid \mathcal{F}_{n-1}]}_{= X_{n-1}} \mid \mathcal{F}_m\big] \overset{\text{repeat}}{= \cdots =} \mathbb{E}[X_m \mid \mathcal{F}_m] = X_m$ $\lrcorner$

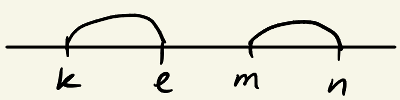
Def Martingale differences:

$D_1 := X_1, \quad D_n := X_n - X_{n-1} \quad$ for $n = 2, 3, \dots$

$\Rightarrow X_n = D_1 + \dots + D_n$

- $\mathbb{E} D_i = 0$ (LTE)
- D_i, D_j are uncorrelated. More generally:

Prop 2. $\forall\, k \le l \le m \le n$: $\quad \mathbb{E}(X_l - X_k)(X_n - X_m) = 0$



$\ulcorner$ $\mathbb{E}\Big[\mathbb{E}\big[\underbrace{(X_l - X_k)}_{\mathcal{F}_m\text{-mble}}(X_n - X_m) \mid \mathcal{F}_m\big]\Big] = \mathbb{E}\Big[(X_l - X_k)\, \underbrace{\mathbb{E}[X_n - X_m \mid \mathcal{F}_m]}_{\shortparallel}\Big] = 0.$

$\underbrace{\mathbb{E}[X_n \mid \mathcal{F}_m]}_{\overset{\shortparallel\,(\text{Prop. }1)}{X_m}} - \underbrace{\mathbb{E}[X_m \mid \mathcal{F}_m]}_{\overset{\shortparallel}{X_m}} = 0$ $\lrcorner$

- Thus, a martingale = sum of uncorrelated mean 0 r.v.'s
  except for D_1