SUBMARTINGALES, SUPERMARTINGALES

Def if, instead of $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$, we have

"$\geq$" $\Rightarrow$ submartingale

"$\leq$" $\Rightarrow$ supermartingale.

Examples: (a) Biased random walk; more generally partial sums $S_n = Y_1 + \cdots + Y_n$ of indep. r.v's with $\mathbb{E}Y_i \geq 0$, is a submartingale

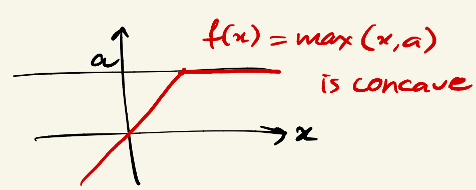
$$\Big[\ \mathbb{E}[\underbrace{S_{n+1}}_{=\,S_n + Y_{n+1}} \mid Y_1, \dots, Y_n] = S_n + \mathbb{E}[Y_{n+1}] \geq S_n\ \Big]$$

(b) $\forall$ convex function $\varphi: \mathbb{R} \to \mathbb{R}$, if (X_n) is a martingale then $(\varphi(X_n))$ is a submartingale

$$\Big[\ \text{Conditional Jensen (p.148)} \Rightarrow \mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] \geq \varphi\big(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]\big) = \varphi(X_n)\ \Big]$$

(c) In particular, if (X_n) is a martingale then

- $|X_n|^p$ is a martingale $\forall p \geq 1$;
- $\max(X_n, a)$ is a supermartingale $\forall a \in \mathbb{R}$



(d) If (X_n) is a martingale and (A_n) is a determistic sequence then $Y_n = X_n + A_n$ is a submartingale

$$\Big[\ \mathbb{E}[\underbrace{Y_{n+1}}_{=\,X_{n+1}+A_{n+1}} \mid \mathcal{F}_n] - \underbrace{Y_n}_{=\,X_n + A_n} = \underbrace{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]}_{=\,X_n} + A_{n+1} - X_n - A_n = A_{n+1} - A_n \geq 0\ \Big]$$

(e) (B_n) needs not be deterministic. It can be random but predictable:

Def A sequence of r.v's (A_n) is predictable if A_n is $\mathcal{F}_{n-1}$-measurable $\forall n$

$\Rightarrow$ can predict "1 day ahead"

Example: A_n = your bet at time n (just before nth spin of the wheel) is based on what happened before (spins $1, \dots, n-1$)

- Example (d-e) is universal:

THM (Doob's decomposition)

- $\forall$ stochastic process $(X_n)_{n=0}^{\infty}$ adapted to a filtration $(\mathcal{F}_n)_{n=0}^{\infty}$ can be decomposed as:
$$X_n = M_n + \underbrace{A_n}_{\text{drift}}$$
where (M_n) is a martingale and (A_n) is predictable with $A_1 = 0$.
- This decomosition is unique (up to a.s.)
- If (X_n) is a submartingale, (A_n) is increasing a.s.

Existence: guided by Example (d), define recursively
$$A_0 := 0; \qquad \boxed{A_{n+1} - A_n = \underbrace{\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]}_{\mathcal{F}_n\text{-mble}} - \underbrace{X_n}_{\mathcal{F}_n\text{-mble}}}, \quad n = 0, 1, 2, \dots \qquad (*)$$

- By induction, A_{n+1} is $\mathcal{F}_n$-mble $\Rightarrow$ predictable.
- Define $M_n := X_n - A_n \Rightarrow \mathcal{F}_n$-mble;
$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - \underbrace{\mathbb{E}[\underbrace{A_{n+1}}_{\mathcal{F}_n\text{-mble}} \mid \mathcal{F}_n]}_{= A_{n+1}} \overset{(*)}{=\!=} X_n - A_n = M_n$$
$\Rightarrow (M_n)$ is a martingale.

- Uniqueness: $\forall$ decomposition as in the statement, take conditional $\mathbb{E} \Rightarrow$
$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \underbrace{\mathbb{E}[M_{n+1} \mid \mathcal{F}_n]}_{\substack{\| \text{(martingale)} \\ M_n = X_n - A_n}} + \underbrace{\mathbb{E}[A_{n+1} \mid \mathcal{F}_n]}_{= A_{n+1} (\mathcal{F}_n\text{-mble})}$$
Rearrange the terms $\Rightarrow$
$$A_{n+1} - A_n = \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] - Y_n \Rightarrow \text{same as } (*) \Rightarrow \text{uniquely determined.}$$

- If (X_n) is a submartingale $\overset{(*)}{\Longrightarrow} A_{n+1} - A_n \geq 0 \ \forall n$, i.e. increasing. $\square$

- Example: If (Z_n) is a martingale then (Z_n^2) is a submartingale $\Rightarrow$
$\exists$ increasing predictable sequence $A_n =: \langle Z_n \rangle$ s.t $\boxed{Z_n^2 - \langle Z \rangle_n}$ is a martingale.

- $\boxed{\text{Prop } \langle Z \rangle_n = \sum_{i=1}^{n} \mathbb{E}[(Z_i - Z_{i-1})^2 \mid \mathcal{F}_{i-1}]}$ In particular, if $Z_n = \sum_{i=0}^{n} \underbrace{Y_i}_{\text{indep. mean } 0}$ then $\langle Z_n \rangle = \sum_{i=0}^{n} \mathbb{E} Y_i^2$

Proof
$$= \mathbb{E}[Z_i^2 - 2 Z_i Z_{i-1} + Z_{i-1}^2 \mid \mathcal{F}_{i-1}] = \mathbb{E}[Z_i^2 \mid \mathcal{F}_{i-1}] - 2 \underbrace{\mathbb{E}[Z_i \mid \mathcal{F}_{i-1}]}_{= Z_{i-1}} Z_{i-1} + Z_{i-1}^2$$
$$= \mathbb{E}[Z_i^2 \mid \mathcal{F}_{i-1}] - Z_{i-1}^2 \overset{(*)}{=} \langle Z \rangle_i - \langle Z \rangle_{i-1}. \text{ Sum up} \Rightarrow \square$$

- $\langle Z \rangle_n$ = "predictable quadratic variation" of (Z_n). W/o $\mathbb{E}[\cdot \mid \mathcal{F}_{i-1}]$, just "quadratic variation" $[Z]_n$.

MARTINGALE TRANSFORM

Prop Let $(X_n)_0^\infty$ be a martingale, $(H_n)_0^\infty$ be a predictable sequence. Then

$$(H \circ X)_n := \sum_{i=1}^{n} H_i \underbrace{(X_i - X_{i-1})}_{\text{martingale differences}} ,$$

is a martingale.

Proof

$$\mathbb{E}\left[(H\circ X)_{n+1} \mid \mathcal{F}_n\right] = (H\circ X)_n + \mathbb{E}\left[\underbrace{H_{n+1}}_{\mathcal{F}_n\text{-meas}}(X_{n+1}-X_n) \mid \mathcal{F}_n\right]$$

$$= (H\circ X)_n + \underbrace{H_{n+1}\Big(\underbrace{\mathbb{E}[X_{n+1}\mid\mathcal{F}_n]}_{= X_n} - \underbrace{\mathbb{E}[X_n \mid \mathcal{F}_n]}_{= X_n}\Big)}_{= 0} = 0 . \quad \square$$

Example: a gambling strategy

X_n = net amount of money we win at time n if we bet \$1 each time

H_n = betting strategy (bet H_n at time n, e.g. ± 1)

$\Rightarrow (H\circ X)_n$ = our winnings at time n

Prop $\Rightarrow$ In a zero-net game (X_n) no matter how good a betting strategy (H_n), the expected winnings are 0.

"You can't beat the system"

Remark Martingale transform is at the heart of the construction of Itô integral

$$\int_0^T \underset{\uparrow\ \forall\text{ adapted process}}{Z_t}\, d\underset{\uparrow\ \text{Brownian motion}}{B_t} = \lim_{\underset{=\{t_1,\dots,t_n\}\subset[0,T]}{\|\pi_n\|\to 0}} \underbrace{\sum_{i=1}^{n} Z_{t_{i-1}}\left(B_{t_i} - B_{t_{i-1}}\right)}_{= (Z\circ B)_n}$$

STOPPING TIMES

Def A r.variable $T^{\in \mathbb{N}}$ is a stopping time w.r.to a filtration $(\mathcal{F}_n)$ if

$$\{T \le n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$$

Ex $\Updownarrow$

$$\{T = n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$$

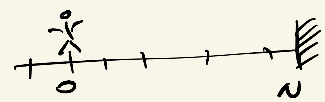
$\left[\underbrace{\{T=n\} = \{T \le n\}}_{\in \mathcal{F}_n} \setminus \underbrace{\{T \le n-1\}}_{\in \mathcal{F}_{n-1} \subset \mathcal{F}_n} \in \mathcal{F}_n \right]$

Ex $\Updownarrow$

$$\{T > n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$$

Examples: (a) constant, e.g. $T = 10$.

(b) (X_n) = symmetric r.walk with 1 wall



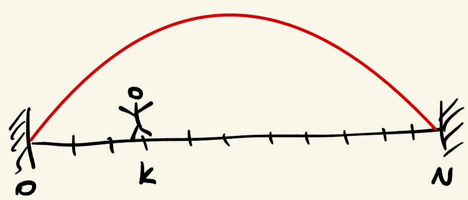
$T := \min\{n : X_n = N\}$

$\mathbb{E}T = \infty$ (even if $N=1$, see Ex. p. 150)

(c) (X_n) = symmetric r.walk with 2 walls, starting at k:

$T := \min\{n : X_n \in \{0, N\}\}$.

$\mathbb{E}T < \infty$ a.s. (why?)



Let's compute it by conditioning on 1st step:

$$\underbrace{\mathbb{E}T}_{\overset{!!}{t_k}} = \underbrace{\mathbb{E}[T|L]}_{= t_{k-1}+1} \cdot \underbrace{P(L)}_{= 1/2} + \underbrace{\mathbb{E}[T|R]}_{= t_{k+1}+1} \cdot \underbrace{P(R)}_{= 1/2}$$

see also [Walsh p. 286]
$\mathbb{E}(X_T^2 - T) = \mathbb{E}(X_0^2 - 0) = k^2$
...

$$\Rightarrow \left\{ \begin{array}{l} t_k = \dfrac{t_{k-1} + t_{k+1}}{2} + 1, \quad k = 1, \dots, N-1 \\ t_0 = t_N = 0 \end{array} \right\} \Rightarrow t_k = \boxed{k(N-k) = \mathbb{E}T}$$

(d) More generally, $\forall$ hitting time: $T = \min\{n : X_n \in B\}$ (martingale: X_n; given set $\subset \mathbb{R}$: B)

(e) Play until either won \$100 or have played 10 games:

$T = T' \wedge 100$ where $T' = \min\{n : X_n = 100\}$

(f) Playing until reach absolute max is NOT a stopping time.

Prop (Stopped martingale) If (X_n) is a martingale and T is a stopping time then $(X_{T\wedge n})$ is a martingale.

$$X_{T\wedge n} - X_0 = \sum_{i=1}^{T\wedge n} (X_i - X_{i-1}) = \sum_{i=1}^{n} \underbrace{\mathbb{1}_{\{i\le T\}}}(X_i - X_{i-1})$$

$\mathcal{F}_{i-1}$-mble, since $\{T\geq i\} = \{T > i-1\} \in \mathcal{F}_{i-1}$
$\Rightarrow$ predictable.

Prop. 169 $\Rightarrow$ $(X_{T\wedge n} - X_1)$ is a martingale.
Since (X_1) is trivially a martingale, $(X_{T\wedge n})$ is a martingale. $\square$

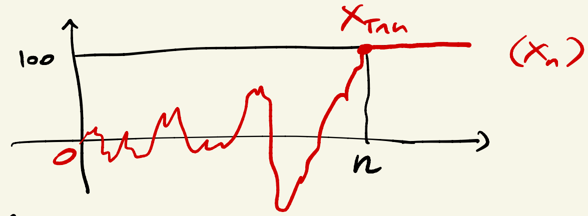
Remark Prop $\Rightarrow$ $\mathbb{E}X_{T\wedge n} = \mathbb{E}X_0 \quad \forall n\in\mathbb{N}$

$\Rightarrow$ NO WINNING STOPPING STRATEGY IN PRESENCE OF A FIXED DEADLINE n

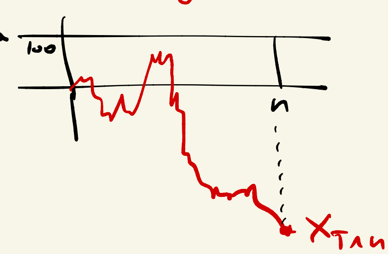
Example: stop when reached \$100:

$$T = \min\{n : X_n = 100\}$$

$$\mathbb{E}X_{T\wedge n} = X_1 = 0 \quad \text{HOW?!}$$



- If $T\leq n$, goal is reached by deadline n;
  $\Rightarrow X_{T\wedge n} = 100$
- But if $T > n$, goal is NOT reached by deadline →
  $X_{T\wedge n}$ can be very negative (debt)



These two counterweigh each other.

Ex (Optional switching): **HW** (270B 2019 p. 93)