TOWARD RANDOM VARIABLES

RECALL MEASURE THEORY:

i.e. if another σ-alg. Σ contains R then Σ must contain $\sigma(R)$

Def Let Ω be $\forall$ set, and $R \subset \Omega$. The smallest σ-algebra that contains R is called the σ-algebra generated by R, as is denoted

$$\sigma(R) = \bigcap_{\Sigma : \Sigma \supset R} \Sigma .$$

all σ-algebras Σ that contain R

Examples:

① $\Omega = \{1,2,3,4\}$ $R = \{\{1,2\},\{3\},\{4\}\}$ $\Rightarrow$ $\sigma(R) = \{\{1,2\},\{3\},\{4\},\{3,4\},\{1,2,3\},\{1,2,4\},\emptyset,\Omega\}$

② Borel σ-algebra: $\Omega = \mathbb{R}^n$, $R = \{\text{open sets}\}$

$\hookrightarrow$ $\mathcal{B} := \sigma(\{\text{open sets}\}) = \sigma(\{\text{intervals } (-\infty, x] \ \forall x \in \mathbb{R}\})$.

③ Product σ-algebra: if $(\Omega, \mathcal{F})$ and (S, Σ) are measurable spaces,

$\hookrightarrow$ $\mathcal{F} \times \Sigma := \sigma\left(F \times G : F \in \mathcal{F}, G \in \Sigma\right)$ is a σ-algebra on $\Omega \times S$.

i.e. $(\Omega \times S, \mathcal{F} \times \Sigma)$ is a measurable space.

④ Infinite product: If $(\Omega_t, \mathcal{F}_t)$, $t \in T$ are measurable spaces,

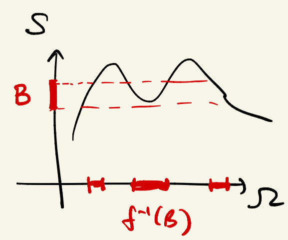
$\prod_{t \in T} \mathcal{F}_t := \sigma(\text{cylinders})$ is a σ-alg on $\prod_{t \in T} \Omega_t$, where a cylinder $= \prod_{t \in T} F_t$ where $F_t \in \mathcal{F}_t$ and $F_t = \Omega$ for all but finitely many $t \in T$.

"cylinder σ-algebra"

Def Let $(\Omega, \mathcal{F})$ and (S, Σ) be measurable spaces.

A map $f : \Omega \to S$ is called a measurable function if

$$f^{-1}(B) \in \mathcal{F} \quad \forall \quad B \in \Sigma \qquad (*)$$



- Key example: $(S, \Sigma) = (\mathbb{R}, \mathcal{B})$

Borel

- Instead of checking that

$$f^{-1}(B) \in \mathcal{F} \quad \forall \ B \in \mathcal{B}, \qquad (*)$$

it is enough to check that

$$\boxed{f^{-1}\left((-\infty, x]\right) \in \mathcal{F} \quad \forall x \in \mathbb{R}} \qquad (**)$$

Assume $(*)$ holds. Consider
$\Sigma := \{B \in \mathcal{B} : f^{-1}(B) \in \mathcal{F}\}$
Then Σ is a σ-algebra (check!),
and Σ contains all intervals $(-\infty, x]$ (by $(*)$)
$\Rightarrow$ by minimality, Σ contains all Borel sets. $\Rightarrow$ $(*)$ holds

SKIP

Prop (Composition) — BRIEFLY:

Assume: $g_1, \dots, g_n : (\Omega, \mathcal{F}) \to (S, \mathcal{F})$ and $f : (S^n, \Sigma^n)$ are measurable. Then:

product space (prev. page)

1. $F(\omega) := f(g_1(\omega), \dots, g_n(\omega))$ is measurable

$$\begin{tikzcd}
(\Omega_1, \mathcal{F}_1) \arrow[dr, "g_1"] & & \\
(\Omega_2, \mathcal{F}_2) \arrow[r, "g_2"] & (S, \Sigma) \arrow[r, "f"] & (R, \mathcal{R}) \\
\cdots & & \\
(\Omega_n, \mathcal{F}_n) \arrow[uur, "g_n"'] & &
\end{tikzcd}$$

2. $\inf g_n$, $\sup g_n$, $\liminf g_n$, $\limsup g_n$ are all measurable.

3. The set $\{\omega \in \Omega : \underbrace{\lim g_n(\omega) \text{ exists}}\}$ is measurable

$\Updownarrow$

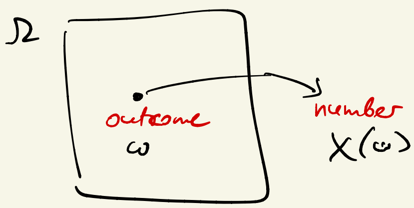
$\limsup g_n(\omega) - \liminf g_n(\omega) = 0$

RANDOM VARIABLES

Def Any measurable function $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ is called a random variable.

← Borel

("random vector" if $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}^n)$).



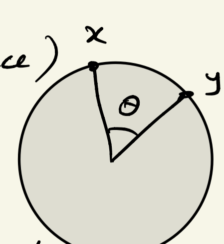
Ex (a) Flip a coin 3 times; $X = \#$ heads

(b) Flip a coin until first T, $X = \#$ flips (wait time)

(c) lifetime of a new phone

(d) Random permutations: return N exams to N students,
$X = \#$ students who receive their own exam

(e) Break a stick at a random pt;
$X = \text{length(longer piece)} - \text{length(shorter piece)}$



(f) Pick 2 pts on a unit circle; $\theta = $ angle

(e) An (imperfect) measurement of ocean's temperature.

Notation: $\{\omega \in \Omega : X(\omega) \in B\} = X^{-1}(B) = \{X \in B\}$ ← will downplay ω, Ω

Def $X \overset{a.s.}{=} Y$ if $P\{\omega : X(\omega) = Y(\omega)\} \left(= P\{X = Y\}\right) = 1$

THE DISTRIBUTION OF A RANDOM VARIABLE

Def Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.

The distribution, or the law of X, is the probability measure $\mathcal{L}_X$ on $(\mathbb{R}, \mathcal{B})$ defined by

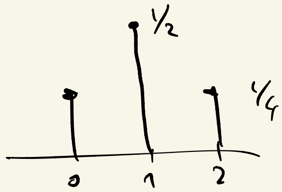
$$\mathcal{L}_X(B) := \mathbb{P}\{X \in B\}, \qquad B \in \mathcal{B}$$

(it's a probability measure indeed – check!)

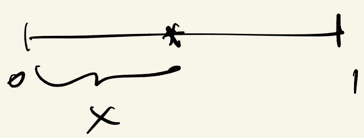
- $\mathcal{L}_X$ encodes what values X takes and their probabilities.

Ex (a) Flip a coin twice, $X = \#$ H's

$\mu(\{0\}) = \frac{1}{4}$, $\mu(\{1\}) = \frac{1}{2}$, $\mu(\{2\}) = \frac{1}{4}$



(b) Break a stick at a random pt,



$\mu(B) = \text{meas}(B \cap [0,1]) \Rightarrow$

$\mu \sim \text{Unif}\,[0,1]$

see before

Def The distribution function of X ("CDF") is the distribution function of $\mathcal{L}_X$, i.e.

$$F(x) = \mathcal{L}_X((-\infty, x]) = \mathbb{P}\{X \le x\}$$

Prop 1. If X is a r.v, F is right-continuous, $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

2. If F satisfies this then $\exists$ r.v. X whose CDF is F

$\ulcorner$ (1) is from before

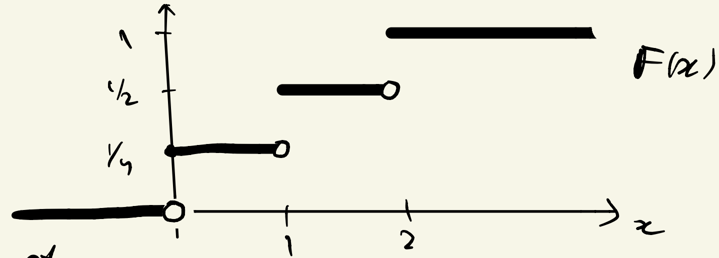
(2) from before, $\exists$ prob. meas $\mathbb{P}$ on $(\mathbb{R}, \mathcal{B})$ s.t.

$$\mathbb{P}((-\infty, x]) = F(x) \quad \forall x.$$

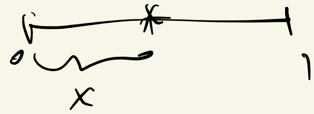
Define X on $(\mathbb{R}, \mathcal{B}, \mathbb{P})$ by $\quad X(\omega) := \omega, \quad \omega \in \mathbb{R}$.

$\Rightarrow F(x) = \mathbb{P}((-\infty, x]) = \mathbb{P}(\{\omega \in \mathbb{R} : \overset{\shortparallel}{\omega} \le x\}) = \mathbb{P}\{X \le x\}$. $\lrcorner$

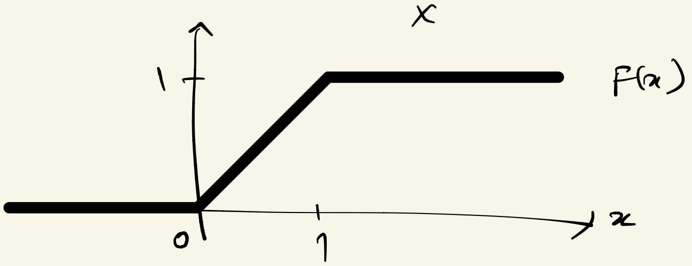
Ex (a) Flip a coin twice, $X = \#$ H's



(b) Break a stick at a random pt



$$P\{X \le x\} = x$$



Uniform distribution on $(0,1)$

$X \sim \mathrm{Unif}\,(0,1)$.

Ex: How to generate a r.v. X with $\forall$ given distribution (given CDF $F(x)$) from $U \sim \mathrm{Unif}\,[0,1]$?

- Assume F is one-to-one for simplicity.

$$X := F^{-1}(U).$$

$$\Rightarrow P\{X \le x\} = P\{F^{-1}(U) \le x\} = P\{U \le F(x)\} = F(x). \quad \square$$

Def (Identical distr.): $X \overset{distr}{=\!=} Y$ if $\mathcal{L}_X = \mathcal{L}_Y$ $\iff P\{X \in B\} = P\{Y \in B\}$ $\forall$ Borel B

$\Updownarrow$

$F_X = F_Y$ pointwise

Note: if $X \overset{a.s.}{=\!=} Y$ then $X \overset{distr}{=\!=} Y$ but not vice versa.

(Example: flip coin 10 times. $X = \#$ heads, $Y = \#$ tails

$X \overset{distr}{=\!=} Y$ by symmetry but $X \overset{a.s.}{\neq} Y$)