Without fixed deadline?

OPTIONAL STOPPING

- let (X_n) be a martingale and T be a stopping time.

Q: is $\mathbb{E}X_T > \mathbb{E}X_0$ possible? (stopping strategy for making money)

Ans: sometimes. Examples:

(a) Symmetric random walk with sticky wall at N,

i.e. $T = \min\{n : X_n = N\}$.

$X_T = N$, $X_0 = 0$ deterministically $\Rightarrow \mathbb{E}X_T > \mathbb{E}X_0$

However, $\mathbb{E}T = \infty$ in this case.

Can one have $\mathbb{E}T < \infty$? Yes:

(b) St. Petersburg martingale $X_0 = 1$, $X_n | X_{n-1} = \begin{cases} 2X_{n-1}, & \text{prob } 1/2 \\ 0, & \text{prob } 1/2 \end{cases}$

Wait until broke: $T = \min\{n : X_n = 0\}$

$\Rightarrow X_T = 0$, $X_0 = 1$ deterministically

$T \sim \text{Geom}(1/2) \Rightarrow \mathbb{E}T = 2$.

However, this game is too volatile: increasing bets.

THM (Doob's optional stopping thm)

Let (X_n) be a martingale and T be a stopping time.

Assume $\exists\, C>0$ s.t. at least one of the following conditions hold:

(i) $|X_{T\wedge n}|<C$ a.s. $\forall n$ ← deadline

(ii) $T<C$ a.s. ← finite lifetime

(iii) $\mathbb{E}T<\infty$ and $\mathbb{E}\big[|X_{n+1}-X_n| \,\big|\, \mathcal{F}_n\big]<C$ a.s. $\forall n$ ← limit on bets

Then $\mathbb{E}X_T=\mathbb{E}X_0$.

Proof (i) $X_{T\wedge n}=X_T \;\; \forall n\geq T \Rightarrow X_{T\wedge n}\to X_T$ a.s $(n\to\infty)$

$$\left.\begin{array}{l} \text{D.C.T} \Rightarrow \mathbb{E}X_{T\wedge n} \to \mathbb{E}X_T \\ \text{stopped mtgale} \;\|\; (\text{p.171}) \\ \qquad \mathbb{E}X_0 \end{array}\right\} \Rightarrow \mathbb{E}X_T=\mathbb{E}X_0. \qquad \square$$

(ii) $X_{T\wedge n}=X_0+\sum_{i=1}^{T\wedge n}(X_i-X_{i-1})$

$\Rightarrow |X_{T\wedge n}|\leq |X_0|+\sum_{i=1}^{T}|X_i-X_{i-1}| =: M$

- Since $T<C$ a.s, this sum has at most C terms, all integrable

  $\Rightarrow \mathbb{E}M<\infty$.

- Thus $(X_{T\wedge n})$ is dominated by an integrable r.v. M.

  Finish as in (i).

(iii) $M=|X_0|+\sum_{i=1}^{\infty}\mathbb{1}_{\{i\leq T\}}|X_i-X_{i-1}|$.

$$\Rightarrow \mathbb{E}M\leq \mathbb{E}|X_0|+\sum_{i=1}^{\infty}\mathbb{E}\Big[\mathbb{E}\big[\underbrace{\mathbb{1}_{\{T\geq i\}}}_{=\,\mathbb{1}_{\{T>i-1\}} \text{ is } \mathcal{F}_{i-1}\text{-mble}}|X_i-X_{i-1}| \,\big|\, \mathcal{F}_{i-1}\big]\Big]$$

$$=\mathbb{E}|X_0|+\sum_{i=1}^{\infty}\mathbb{E}\Big[\mathbb{1}_{\{T\geq i\}}\underbrace{\mathbb{E}\big[|X_i-X_{i-1}| \,\big|\, \mathcal{F}_{i-1}\big]}_{\leq\, C}\Big]$$

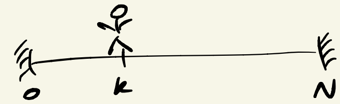
$$\leq \mathbb{E}|X_0|+C\underbrace{\sum_{i=1}^{\infty}\mathbb{P}\{T\geq i\}}_{=\,\mathbb{E}T<\infty \text{ (integrated tail formula)}}$$

$<\infty$.

Finish as before.

APPLICATIONS

Ⓐ Gambler's Ruin : $P(\text{reach } N \text{ before } 0) = ?$ (start at k)



Sym. random walk with 2 walls. O.S.T.(i) applies $\Rightarrow$ for $T = \min\{n : X_T \in \{0,N\}\}$

$$\underbrace{EX_T}_{\substack{\| \\ N \cdot P\{X=N\}}}^{X\in\{0,N\}} = EX_0 = k \implies \boxed{P\{X=N\} = \frac{k}{N}}$$

We recovered the result of Gambler's Ruin problem (p. 146)

Ⓑ Expected hitting time of $\{0,N\} = ?$

Consider the quadratic martingale $(X_n^2 - n)$,

apply O.S.T. (i) $(|X_{T\wedge n}| \le N \ \forall n)$

$$\Rightarrow \mathbb{E}\left[X_T^2 - T\right] = \mathbb{E}\left[X_0^2 - 0\right] = k^2 \quad \Rightarrow \quad \mathbb{E}[T] = \underbrace{\mathbb{E}[X_T^2]}_{\substack{Ⓐ \| \\ N^2 \cdot \frac{k}{N}}} - k^2 = \boxed{k(N-k)}$$

We recovered the result on p. 170.

Ⓑ **Wald's equation**

Let $Z_1, Z_2, \dots$ be iid r.v's with finite mean $\overset{\mu}{\|}$, T be a stopping time with $ET < \infty$. Then

$$\mathbb{E}\left[\sum_{i=1}^{T} Z_i\right] = \mathbb{E}[T] \cdot \mathbb{E}[Z_1]$$

Proof $S_0 := 0$, $S_n := Z_1 + \dots + Z_n \Rightarrow X_n := S_n - n\mu$ is a martingale

Then OST (iii) applies

$$\Big(\mathbb{E}\big[\underbrace{|X_{n+1} - X_n|}_{\substack{\| \\ Z_{n+1} - \mu \text{ is } \mathcal{F}_n\text{-mble}}} \mid \mathcal{F}_n\big] = \mathbb{E}|Z_{n+1} - \mu| \le \mathbb{E}|Z_1| + |\mu| =: C < \infty \Big)$$

$$\Rightarrow \mathbb{E}\underbrace{X_T}_{\substack{\| \\ S_T - T\mu}} = \mathbb{E}X_0 = 0 \quad \Rightarrow \quad ES_T = \mathbb{E}[T]\mu. \qquad \square.$$

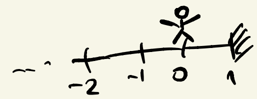
HW: Show by example that independence assumption can't be removed:

$Z_n = Z \sim \text{Rademacher}$; $T = \begin{cases} 2 & \text{if } Z = 1 \\ 1 & \text{if } Z = -1 \end{cases}$

Ⓒ Symmetric random walk. T = time to get from 0 to 1

i.e. $S_n = \sum_{i=1}^{n} Z_i$ ← indep. Rademacher $\qquad T = \min\{n : S_n = 1\}$.

$\boxed{\mathbb{E}T = \infty}$



If $\mathbb{E}T < \infty$ then, by Wald's equation,

$$\mathbb{E}\Big[\sum_{i=1}^{T} Z_i\Big] = \mathbb{E}[T] \cdot \underbrace{\mathbb{E}[Z_1]}_{0} = 0. \quad \text{But } \sum_{i=1}^{T} Z_i = 1 \text{ deterministically. } ↯$$

(iid Rademacher)

We recovered the result on p. 151.

Ⓓ $\boxed{\textbf{Prop (Distribution of } T\textbf{)} \quad \forall n \in \mathbb{N}: \quad P\{T = 2n-1\} = \frac{1}{2n-1}\binom{2n}{n} 2^{-2n}}$ $\approx \frac{1}{2\sqrt{\pi}} n^{-3/2}$, $n \to \infty$

Proof. Product martingale: fix $\forall \theta \in \mathbb{R}$;

$$X_0 := 1, \qquad X_n := \frac{\exp(\theta S_n)}{M(\theta)^n} = \prod_{i=1}^{n} \frac{\exp(\theta Z_i)}{\mathbb{E}\exp(\theta Z_i)}$$

where $M(\theta) = \mathbb{E}\exp(\theta Z_1) = \boxed{\frac{e^{-\theta} + e^{\theta}}{2}}$.
"MGF"

The stopped martingale $X_{T \wedge n}$ is bounded:

$$0 \le X_{T\wedge n} \le \frac{\exp(\theta \cdot 1)}{1^n} = e^{\theta}.$$

- ⇒ O.S.T. (i) applies and yields

$$\left.\begin{array}{c} \mathbb{E}X_T = \mathbb{E}X_0 = 1 \\ \| \\ \mathbb{E}\left[\dfrac{\overbrace{\exp(\theta S_T)}^{=1}}{M(\theta)^T}\right] \end{array}\right\} \Rightarrow \boxed{\mathbb{E}\left[M(\theta)^{-T}\right] = e^{-\theta}} \qquad (*)$$

- Generating function $\mathbb{E}[s^T] = ?$ $\forall s > 0$.

Change var's to $\frac{1}{s} = M(\theta) = \frac{e^{-\theta} + e^{\theta}}{2}$ varies in $[1, \infty)$

$\Rightarrow s$ varies in $(0, 1]$.

Solving $\Rightarrow$ $e^{-\theta} = \frac{1 \pm \sqrt{1-s^2}}{s}$

$(*) \Rightarrow \mathbb{E}[s^T] = \frac{1 \pm \sqrt{1-s^2}}{s}$, $s \in (0,1]$

Considering $s \downarrow 0$ we see that the "$-$" sign is valid $\Rightarrow$

$$\mathbb{E}[s^T] = \frac{1-\sqrt{1-s^2}}{s}, \quad s \in (0,1]$$

$$\sum_{k=1}^{\infty} s^k P\{T=k\} \qquad \text{Taylor series } (s\downarrow 0) \qquad \sum_{n=1}^{\infty} \frac{1}{2n-1}\binom{2n}{n} 2^{-2n} \cdot s^{2n-1}$$

Matching the coefficients $\Rightarrow$ $\square$.

- Combinatorial proof ?

- Martingales in continuous time.