$\mathbb{E}X$ in terms of CDF (kcw):

Prop (Integrated tail formula) $\forall$ nonnegative r.v. X:

$$\mathbb{E}X = \int_0^\infty \mathbb{P}\{X \geq t\}\,dt \underset{\uparrow}{=} \sum_{k=1}^\infty \mathbb{P}\{X \geq k\}$$

if X is integer-valued

Application: the longest increasing subsequence in a random permutation.

$$\pi: \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \downarrow & \downarrow & \cdots \\ 3 & 10 & 4 & 7 & 2 & 6 & 1 & 9 & 8 & 5 \end{array}$$

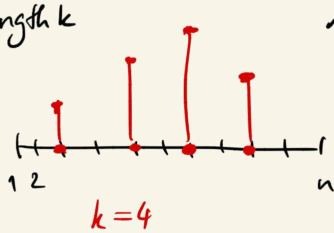
longest increasing subsequence. length $L(\pi) = 4$

THM $c\sqrt{n} \leq \mathbb{E}L(\pi) \leq C\sqrt{n}$ (following prob-examples-cool.pdf)

Proof Upper bound $\forall k = 1, \dots, n$:

$$\mathbb{P}\{L(\pi) \geq k\} \leq \mathbb{P}\{\exists \text{ subsequence of length } k : \text{ increasing}\} = \mathbb{P}\Big(\bigcup_{\substack{\text{subsequences} \\ \text{of length } k}} \{\text{subseq. is increasing}\}\Big)$$

$$\underset{\text{union bound}}{\leq} \sum_{\substack{\text{subsequences} \\ \text{of length } k}} \mathbb{P}\{\text{subseq. is increasing}\} = \binom{n}{k} \cdot \frac{1}{k!}$$



Stirling: $k! \geq (k/e)^k$

$$\leq \frac{n^k}{k!} \cdot \frac{1}{k!} \leq \left(\frac{e\sqrt{n}}{k}\right)^{2k}.$$

This bound is trivial for $k \leq e\sqrt{n}$. Will use it for $k \geq \lceil 2e\sqrt{n} \rceil$ where $\dots \leq \frac{1}{4^k}$

$$\mathbb{E}L(\pi) \underset{\text{integrated tail}}{=} \sum_{\ell=1}^n \mathbb{P}\{L(\pi) \geq k\} \leq \lceil 2e\sqrt{n} \rceil + \underbrace{\sum_{k \geq \lceil 2e\sqrt{n} \rceil} \frac{1}{4^k}}_{\leq 1} \leq C\sqrt{n}.$$

Lower bound follows by symmetry from

Erdős-Szekeres Thm (1935):

$\forall$ given k, ℓ $\forall$ sequence of distinct real numbers with length at least $(k-1)(\ell-1)+1$ has either an increasing subsequence of length k, or a decreasing subsequence of length ℓ.

(proof by pigeonhole principle – Wiki)

$\Rightarrow \forall$ permutation has either increasing or decreasing subsequence of length $c\sqrt{n}$.

$\Rightarrow L(\pi) + D(\pi) \geq c\sqrt{n}$ a.s. ($D(\pi)$: longest decreasing) But $L(\pi)$, $D(\pi)$ have the same distribution (symmetry)

$$\Downarrow$$

$$\mathbb{E}L(\pi) + \underbrace{\mathbb{E}D(\pi)}_{= \mathbb{E}L(\pi)} \geq c\sqrt{n} \Rightarrow \mathbb{E}L(\pi) > c\sqrt{n}/2 \quad \square$$

Baik-Deift-Johansson theorem (1999)

$$\mathbb{P}\left\{\frac{L(\pi) - 2\sqrt{n}}{n^{1/6}} \leq t\right\} \to \text{CDF(Tracy-Widom distr.)}$$

(ABSOLUTELY) CONTINUOUS DISTRIBUTIONS

Recall from measure theory:

THM (Radon-Nikodym Thm)

Let (S,Σ) be a measurable space.

Let μ, λ be two σ-finite measures on (S,Σ) (S = countable union of mble sets with finite measures)

Assume $\mu \ll \lambda$ (absolutely continuous) ($\lambda(A)=0 \Rightarrow \mu(A)=0 \;\; \forall A \in \Sigma$)

Then $\exists$ measurable function $f: S \to [0,\infty)$ ("density")

s.t. $\mu(B) = \int_B f \, d\lambda \quad \forall \; B \in \Sigma$ $\quad$ $:= \int f \mathbb{1}_B \, d\lambda$

- $\mu(B) = \int_B 1 \, d\mu$ "$\Rightarrow$" $f\,d\lambda = 1\,d\mu$ "$\Rightarrow$" $\boxed{f = \frac{d\mu}{d\lambda}}$ "Differentiation"
- More generally: $\forall$ measurable function $g: S \to \mathbb{R}$:

$$\boxed{\int g \, d\mu = \int g f \, d\lambda}$$ "Change of variable"

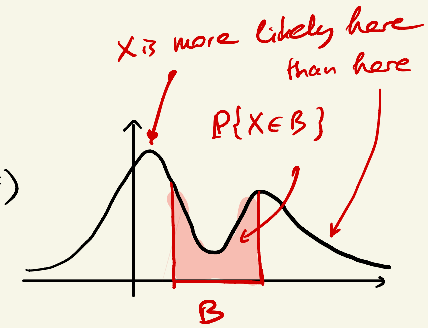
[For indicators, this is RNT $\Rightarrow$ follow def of Lebesgue integral.] – DIY

Def A random variable X has an (absolutely) continuous distribution if its distribution $\mathcal{L}_X$ is absolutely continuous w.r.to the Lebesgue measure on $\mathbb{R}$.

RNT $\Rightarrow$ $\exists$ density of X: $\quad f := \frac{d\mathcal{L}_X}{d\lambda}$ ("pdf", probability density function of X)

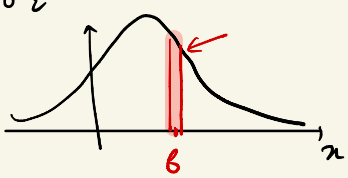
- Probabilities in terms of pdf: $\forall$ Borel $B \subset \mathbb{R}$:

$$\mathbb{P}\{X \in B\} \overset{def}{=} \mathcal{L}_X(B) \overset{RNT}{=} \int_B f \, d\lambda \text{ (Lebesgue)}, \text{ i.e. } \boxed{\mathbb{P}\{X\in B\} = \int_B f(x)\,dx} \quad (*)$$



- Intuitive meaning of density: Lebesgue differentiation thm: take $B = [b-\varepsilon, b+\varepsilon] \Rightarrow$

$$f(b) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{b-\varepsilon}^{b+\varepsilon} f(x)\,dx = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \mathbb{P}\{|X-b| \le \varepsilon\}, \text{ i.e.}$$



$$\boxed{\mathbb{P}\{|X-b| \le \varepsilon\} \approx 2\varepsilon \cdot f(b).}$$ for small $\varepsilon > 0$.

• CDF $\leftrightarrow$ PDF : For $B=(-\infty,x]$, (*) gives

$$\boxed{F(x)=\int_{-\infty}^{x} f(y)\,dy} \xrightarrow{\text{differentiate}} \boxed{f(x)=F'(x)}$$

wherever F' exists, which is a.e. by Lebesgue differentiation theorem

$\Rightarrow$ the distribution of X is determined by pdf, pmf, or cdf.

• EX in terms of pdf : $EX \overset{\text{p.19}}{=} \int_{-\infty}^{\infty} x\,d\mathcal{L}_X \overset{\text{RNT}}{=} \boxed{\int_{-\infty}^{\infty} x f(x)\,dx}$ ← analogous to $EX=\sum_i x_i P\{X=x_i\}$ for discrete r.v.

More generally, $\boxed{Eg(X)=\int_{-\infty}^{\infty} g(x) f(x)\,dx}$

Prop (Characterization of pdf)

1. $\forall$ r.v. X with absolutely continuous distribution, the pdf f of X is an integrable function s.t.
$$f \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\,dx = 1. \qquad (\circ)$$
2. $\forall$ integrable function $f: \mathbb{R}\to\mathbb{R}$ that satisfies $(\circ)$ is the pdf of some r.v. X.

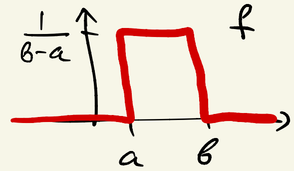
Proof 1. $f\geq 0$ by RNT, $\int_{-\infty}^{\infty} f(x)\,dx \overset{(*\text{p.25})}{=} P\{X\in(-\infty,\infty)\}=1$.

2. Define
$$\mu(B) := \int_B f\,d\lambda \quad \forall \text{ Borel } B. \qquad (\circ\circ)$$
Then μ is a probability measure on $(\mathbb{R},\mathcal{B})$ (by properties of Lebesgue meas) DIY

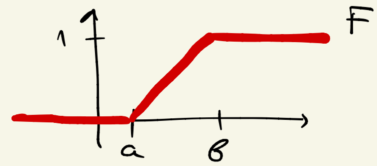
Now let $X(x) := x$ on $(\mathbb{R},\mathcal{B},\mu)$

$\Rightarrow$ $\mathcal{L}_X(B) = \mu(B) \overset{(\circ\circ)}{=} \int_B f\,d\lambda \Rightarrow f$ is the pdf of X. $\square$

Example (a) $X \sim \mathrm{Unif}[a,b] \Rightarrow$



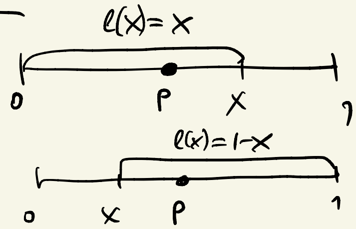
pdf



cdf

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\cdot \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}.$$

(b). A stick of unit length is broken into 2 pieces at random. Compute the expected length of the piece that contains a given pt $p \in [0,1]$



$X \sim \mathrm{Unif}[0,1]$ breaking pt.

$$\ell(X) = \begin{cases} X & \text{if } X > p \\ 1-X & \text{if } X \le p \end{cases}$$

$$\mathbb{E}\,\ell(X) = \int_{-\infty}^{\infty} \ell(x) f(x)\,dx = \int_0^1 \ell(x)\,dx = \int_0^p (1-x)\,dx + \int_p^1 x\,dx = \frac{1}{2} + p(1-p).$$