Homework 1
Probability: a Graduate Course

---

1. Expectation minimizes the squared error

Let X be a random variable with finite expectation. Show that the function $f(t) = \mathbb{E}(X-t)^2$ is minimized at $t = \mathbb{E}\, X$.

2. Median minimizes the absolute error

Let X be a random variable with continuous distribution and finite expectation. Show that the function $g(t) = \mathbb{E}|X - t|$ is minimized at $t = M(X)$.

3. Stein's lemma ("Gaussian integration by parts")

Let $Z \sim N(0,1)$, i.e. Z is a standard normal random variable. Let $g : \mathbb{R} \to \mathbb{R}$ be a function. Show that

$$\mathbb{E}\, g'(Z) = \mathbb{E}\left[Z\, g(Z)\right].$$

Feel free to make any reasonable regularity assumptions on the function g, such as continuous differentiability, integrability, behavior at $\pm\infty$, etc.

4. Gaussian moments

Let $Z \sim N(0,1)$ and $n \in \mathbb{N}$.

(a) Using Stein's lemma, check that

$$\mathbb{E}[Z^{n+1}] = n\, \mathbb{E}[Z^{n-1}].$$

(b) Find a formula for $\mathbb{E}[Z^n]$ that involves only n.

5. Maximum of gaussians

Consider $n > 1$ and random variables $Z_1, \ldots, Z_n \sim N(0,1)$, which may or may not be independent. Show that

$$\mathbb{E} \max_{i=1,\ldots,n} |Z_i| \le C\sqrt{\log n},$$

where C is an absolute constant.

Hint: Bound $\mathbb{E} \max_i |X_i|$ *by* $\mathbb{E}\left(\sum_{i=1}^n |X_i|^p\right)^{1/p}$*; move the expected value inside the sum (how?); use Problem 4 to bound the moments; finally optimize in* p *or just guess any value of* p *that works.*

6. Integrated tail formula

(a) Let X be a random variable that takes on nonnegative values. Prove that

$$\mathbb{E}\, X = \int_0^\infty \mathbb{P}\left\{X > t\right\}\, dt.$$

(b) Let X be a random variable that takes on nonnegative integer values. Prove that

$$\mathbb{E}\, X = \sum_{k=0}^\infty \mathbb{P}\left\{X > k\right\}.$$

7. Gibbs measure

Consider a finite set Ω ("state space"), a positive number T ("temperature"), and a function $E : \Omega \to \mathbb{R}$ (which computes the "free energy" of the system at each state). The Gibbs measure on Ω is first defined for single-element subsets by

$$P(\{\omega\}) = \frac{1}{Z} \exp\left(-\frac{E(\omega)}{T}\right), \quad \omega \in \Omega,$$

where $Z = \sum_{\eta\in\Omega} \exp(-E(\eta)/T)$ is a normalization constant ("partition function"). Then Gibbs measure is extended to all subsets of Ω by additivity. You will now prove that if the temperature T increases to infinity, the system becomes chaotic and can be found in any state with the same probability. Conversely, if the temperature decreases to zero, the system freezes in any lowest-energy state.

(a) Prove that if $T \to \infty$, then P converges to the uniform measure on Ω, i.e.

$$P(\{\omega\}) \to \frac{1}{|\Omega|} \quad \text{for all } \omega \in \Omega.$$

(b) Prove that if $T \to 0$, then P converges to the uniform measure on the subset Ω_0, which consists of all states $\omega_0 \in \Omega$ such that $E(\omega_0) = \min_{\omega\in\Omega} E(\omega)$.

8. Union bound

Every sequence of events $E_1, E_2, \ldots$ satisfies the inequality

$$\mathbb{P}\left(\bigcup_{i=1}^\infty E_i\right) \le \sum_{i=1}^\infty \mathbb{P}(E_i).$$

This fact, known as "union bound" or Boole's inequality, was proved in class using integrals. Prove it directly, without using integration or expectation.

Try to express the event $E = \bigcup_{i=1}^\infty E_i$ as a disjoint union of certain events.

9. Inclusion-exclusion principle

Prove the inclusion-exclusion principle for arbitrary number of events $E_1, \dots, E_n$ by taking expectation of a product of certain random variables.

Hint: we essentially did this for $n = 2$ in class.

10. Generating a random variable with a given distribution

Demonstrate how to transform a random variable U uniformly distributed on $[0, 1]$ into a random variable X with a given cumulative distribution function F.

In class, we did this under the assumption that F is strictly increasing. To remove this assumption, you will need to define a kind of "right inverse" of F.