Homework 2
Probability: a Graduate Course

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1. Get me out of Wyoming

Wyoming is (approximately) an $R \times R$ square where $R \approx 313$ miles. Choose a point P in Wyoming uniformly at random. Compute the expected distance from P to the closest point on Wyoming's border.

2. Absolute continuity: joint vs individual

Find two absolutely continuous random variables X and Y that are not jointly absolutely continuous.

3. Finite expectation, infinite variance?

Does there exist a random variable X satisfying the following three properties:

(i) $X \geq 0$ almost surely;
(ii) $\mathbb{E}\, X = 1$;
(iii) $\mathrm{Var}(X) = \infty$?

4. Jensen's inequality is usually strict

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly convex function. Let X be a random variable such that $\mathbb{E}\,|X| < \infty$ and $\mathbb{E}\,|\varphi(X)| \leq \infty$. Show that

$$\varphi(\mathbb{E}\, X) = \mathbb{E}(\varphi(X)) \quad \text{implies} \quad X = \mathbb{E}\, X \text{ a.s.}$$

5. An algorithm to compute pairwise independent random variables

Let $p \geq 3$ be a prime. Let X and Y be independent random variables that are uniformly distributed on $\{0, \ldots, p-1\}$. Define

$$Z_n := (X + nY) \mod p, \quad n = 0, \ldots, p-1.$$

Show that the random variables Z_n are pairwise independent, but not jointly independent.

6. Chebyshev's inequality is optimal... Is it?

(a) For any given $\mu \in \mathbb{R}$, $\sigma > 0$, $k > 0$, show that there exists a random variable X with mean μ and variance σ^2 and for which Chebyshev's inequality becomes an equality:

$$\mathbb{P}\left\{|X-\mu| \geq k\sigma\right\} = \frac{1}{k^2}.$$

(b) Show that for any random variable X with mean μ and variance σ^2, one has

$$\mathbb{P}\left\{|X-\mu| \geq k\sigma\right\} = o\Big(\frac{1}{k^2}\Big) \quad \text{as } k \to \infty.$$

Why do parts (a) and (b) not contradict each other?

7. Pairwise independence

By definition, events E, F, G are independent if all of the following four equations hold:

$$\begin{aligned} \mathbb{P}(E \cap F) &= \mathbb{P}(E)\,\mathbb{P}(F), \\ \mathbb{P}(E \cap G) &= \mathbb{P}(E)\,\mathbb{P}(G), \\ \mathbb{P}(F \cap F) &= \mathbb{P}(E)\,\mathbb{P}(F), \\ \mathbb{P}(E \cap F \cap G) &= \mathbb{P}(E)\,\mathbb{P}(F)\,\mathbb{P}(G). \end{aligned}$$

We gave an example showing that pairwise independence does not necessarily implies independence: it may happen that the first three equations above hold while the last (fourth) equation fails. Can it happen that the *last* three equations hold while the first equation fails?

(I have not thought about this problem before, and am curious to know the answer.)

8. Probability of divisibility

Two numbers X and Y are chosen in $\{1, \ldots, n\}$ randomly, independently and uniformly. We are interested in $p(n) = \mathbb{P}\{X \text{ divides } Y\}$. Find the asymptotic behavior of $p(n)$ as $n \to \infty$.

9. Coalescence

A total of n bar magnets are placed end to end in a line with random independent orientations. Adjacent like poles repel, ends with opposite polarities join to form blocks. Find the expected number of blocks of joint magnets.

10.

Suppose that n points are chosen on the (circumference of a) circle randomly, independently and uniformly. What is the probability that all points lie in some semicircle, like in the figure below?