Homework 4
Probability: a Graduate Course

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1. Reverse Borel-Cantelli?

Let $E_1, E_2, \dots$ be events in the same probability space. Assume that the probability that events E_n occur infinitely often equals 0.

(a) Does it follow that $\sum_{n=1}^{\infty} \mathbb{P}(E_i) < \infty$?

(b) Does it follow that $\lim_{n\to\infty} \mathbb{P}(E_i) = 0$?

Prove or give a counterexample for each statement.

2. Uncorrelated vs. independent

We showed in class that jointly normal random variables are independent if and only if they are uncorrelated. Let us show that *joint* normality is essential here. Give an example of a pair of normal random variables X_1, X_2 on the same probability space, which are uncorrelated but not independent.

3. A projection of a normal distribution

Let Z be a normal random vector in $\mathbb{R}^n$, and let A be a fixed $m \times n$ of rank m, where $m \leq n$. Prove that the random vector AZ is normal. (This does not follow directly from the definition of normal distribution, which requires that $m = n$.)

4. An explicit construction of independent random variables

Let X be a random variable that is uniformly distributed in $[0, 1]$. Show that in the base-2 representation[1] the digits of X are independent $\mathrm{Ber}(1/2)$ random variables.

5. Diophantine approximation

Call number $x \in [0, 1]$ *badly approximable* (by rationals) if there exists $c = c(x) > 0$ and $\varepsilon = \varepsilon(x) > 0$ such that for any $p, q \in \mathbb{N}$ we have

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^{2+\varepsilon}}.$$

Prove that almost all numbers in $[0, 1]$ are badly approximable (i.e. all except a set of Lebesgue measure zero).

[1] For example, the base-2 representation of $X = 0.65$ is $0.10100110\cdots$

(Hint: fix c, ε. For each q, consider the set E_q of numbers x that satisfy the reverse inequality. Use Borel-Cantelli lemma for these sets.)

6. Records

Let $X_1, X_2, \ldots$ be independent random variables. Show that $\sup_n X_n < \infty$ almost surely if and only if there exists $M \in \mathbb{R}$ such that

$$\sum_n \mathbb{P}\{X_n > M\} < \infty.$$

7. Sums of independent random variables

Let $X_1, X_2, \ldots$ be independent random variables. Denote their partial sums by $S_n := X_1 + \cdots + X_n$. Let $a_1, a_2, \ldots$ be a sequence of real numbers that increases to infinity. Show that $\limsup_n S_n/a_n$ is constant almost surely, i.e. there exist a (nonrandom) number $s \in \mathbb{R} \cup \{+\infty\}$ such that

$$\mathbb{P}\left\{\limsup_n \frac{S_n}{a_n} = s\right\} = 1.$$

(Hint: use Kolmogorov's zero-one law.)

8. Everyone has a chance to lead infinitely often

Suppose that we have n boxes. An infinite sequence of balls are dropped into the boxes independently and uniformly at random. Prove that with probability 1, each box has the maximum number of balls among all boxes infinitely often.

9. Independence requires an exponentially large space

Suppose we can find n independent events in some probability space, whose probabilities lie strictly between zero and one. Show that the sample space Ω must have cardinality at least 2^n.

10. Rademacher and Walsh

Let $X_1, \ldots, X_n$ be independent Rademacher random variables. For every subset $I \subset \{1, \ldots, n\}$, consider the product

$$W_I := \prod_{i \in I} X_i.$$

(By convention, set $W_\emptyset := 1$.) The 2^n random variables W_I are called *the Walsh system*. Show that Walsh random variables are uncorrelated but not (jointly) independent.