Homework 5

Probability: a Graduate Course

1. Convergence on discrete spaces

Let $X_1, X_2, \dots$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a countable set and $\mathcal{F} = 2^\Omega$ (the power set). For each statement below, prove or give a counterexample.

(a) $X_n \to X$ in probability if and only if $X_n \to X$ a.s.
(b) $X_n \to X$ in distribution if and only if $X_n \to X$ a.s.

2. WLLN for non-identically distributed r.v.'s

Let $S_n := X_1 + \cdots + X_n$, where $X_1, X_2, \dots$ be independent random variables that satisfy

$$\frac{\operatorname{Var}(X_n)}{n} \to 0 \quad \text{as } n \to \infty.$$

(a) Prove that

$$\frac{S_n - \mathbb{E}[S_n]}{n} \to 0 \quad \text{in probability.}$$

(b) Show that *almost sure* convergence in (a) may fail. Why does not this contradict the strong law of large numbers?

Hints: (a) *Show that* $\operatorname{Var}(S_n/n) \to 0$ *and apply Chebyshev's inequality.* (b) *Let* X_n *take value* n *with probability* $1/(2n \ln n)$, *value* $-n$ *with the same probability, and value* 0 *with the remaining probability.*

3. Keep breaking the stick

Let $X_0 = 1$ and define X_n inductively by choosing X_{n+1} uniformly at random from the interval $[0, X_n]$. Prove that

$$\frac{\ln X_n}{n} \to c \quad \text{a.s.}$$

and find the value of the constant c.

Hint: express X_n *as a product of iid variables, take logarithm, and use the strong law of large numbers.*

4. Failure of LLN

Construct a sequence of independent mean zero random variables $X_1, X_2, \dots$ such that

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to \infty \text{ a.s.}$$

Why does not this example contradict the law of large numbers?

Hint: revisit the St. Petersburg paradox from Homework 4, Exercise 7.

5. SLLN for the number of recurring events

Suppose disasters occur at random times X_i apart from each other. Precisely, n th disaster occurs at time $T_n := X_1 + \cdots + X_n$, where X_i are i.i.d. random variables taking positive values and with finite mean μ. Let

$$N(t) := \max\{n : \ T_n \le t\}$$

be the number of disasters that have occurred by time t. Prove that

$$N(t) \to \infty \quad \text{and} \quad \frac{N(t)}{t} \to \frac{1}{\mu}$$

almost surely as $t \to \infty$.

Hint: check that $N(t) < n$ iff $T_n > t$, and $T_{N(t)} \le t < T_{N(t)+1}$. Use the strong law of large numbers for T_n/n.

6. Kolmogorov's two series theorem is not reversible

Find a sequence of independent mean zero random variables (X_n) for which $\sum_{n=1}^{\infty} X_n$ converges almost surely, yet $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) = \infty$.

Hint: make X_n take large values with tiny probability.

7. Levy's random series theorem

Let (X_n) be a sequence of independent random variables. Prove that the series $\sum_{n=1}^{\infty} X_n$ converges in probability if and only if it converges almost surely.

Hint: modify the proof of Kolmogorov's two series theorem, using Etemadi's maximal inequality.

8. Convergence of normal distributions

Let $\mu_n, \mu \in \mathbb{R}$ and $\sigma_n, \sigma \ge 0$. Let $X_n \sim N(\mu_n, \sigma_n^2)$ and $X \sim N(\mu, \sigma^2)$. Prove that

$$X_n \xrightarrow{d} X \quad \text{if and only if} \quad \mu_n \to \mu \text{ and } \sigma_n^2 \to \sigma^2.$$

9. Convergence to a constant

Let X_n be random variables and c be a constant. Prove X_n converges to c in distribution if and only if X_n converges to c in probability.

Hint: use Portmanteau Lemma.

10. Continuous mapping theorem

Let X_n be random variables and $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. For each mode of convergence – almost sure, in probability, and in distribution – prove the following statement:

$$X_n \to X \quad \text{implies} \quad h(X_n) \to h(X).$$

Hint: For convergence in distribution, use Portmanteau Lemma. For convergence in probability, use truncation and uniform continuity of h on an interval.

11. Convergence in distribution and convergence of means

Let X_n, X be random variables with finite means.

(a) Assume that $\sup_n \mathbb{E}\, X_n^2 < \infty$. Prove that $X_n \xrightarrow{d} X$ implies $\mathbb{E}\, X_n \to \mathbb{E}\, X_n$.
(b) Show by example that the assumption $\sup_n \mathbb{E}\, X_n^2 < \infty$ cannot be removed in general.

Hint: (a) *Use truncation.*

12. Scheffé's lemma

Let X_n, X be random variables.

(a) (For absolutely continuous distributions) Prove that if the probability density functions of X_n converge to the probability density function of X pointwise, then X_n converges to X in distribution.
(b) (For discrete distributions) Prove that if the probability mass functions of X_n converge to the probability mass function of X pointwise, then X_n converges to X in distribution.
(c) (No converse) In general, convergence in distribution does not imply pointwise convergence of probability density functions. Find an example of random variables X_n with densities f_n so that $X_n \xrightarrow{d} X \sim \text{Unif}[0,1]$ but $f_n(x) \not\to 1$ for any $x \in [0,1]$.

Hint: (a) *The densities f_n, f satisfy the triangle inequality $|f_n| + |f| - |f - f_n| \geq 0$. Apply the Fatou lemma to the function in the left hand side.*