Homework 6
Probability: a Graduate Course

1. Increasing functions are positively correlated

Let X be a random variable and $f, g : \mathbb{R} \to \mathbb{R}$ be nondecreasing functions. Prove that random variables $f(X)$ and $g(X)$ are non-negatively correlated. Feel free to add any reasonable integrability assumptions.

Hint: Let Y be an independent copy of X. Check that that the random variables $f(X) - f(Y)$ and $g(X) - g(Y)$ always have the same sign. Take expectation of their product.

2. Extreme values

Let X_i be i.i.d. random variables each having exponential distribution with mean 1, i.e. $\mathbb{P}\{X_i > x\} = e^{-x}$ for all $x \in \mathbb{R}$. Consider

$$M_n \coloneqq \max_{i \le n} X_i.$$

Show that $M_n - \log n$ converges in distribution to the random variable Y with the standard Gubmel distribution, i.e. the random variable with cdf $F(x) = \exp(-e^{-x})$.

Hint: compute the cdf of M_n and take the limit.

3. Slutsky's theorem

Consider the following statement:

$$\text{if } X_n \to X \text{ weakly and } Y_n \to Y \text{ weakly then } X_n + Y_n \to X + Y \text{ weakly.} \tag{1}$$

(a) Find an example showing that (1) is false in general.
(b) Prove that if Y is a constant, then (1) is true.
(c) Prove that if X_n and Y_n are independent, and X and Y are independent, then (1) is true.

Hint for (b): Let $Y = 0$. By Portmanteau lemma, it is enough to prove that $\mathbb{E}\, Z_n \to 0$ where $Z_n \coloneqq h(X_n + Y_n) - h(X_n)$ and h is any function that is uniformly bounded along with its derivative. Calculus should help you bound Z_n when Y_n is small.

4. Sums of independent Poissons

Let $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$ be independent. Use characteristic functions to check that $X + Y \sim \text{Pois}(\lambda + \mu)$.

5. The rate of convergence in CLT

The nonasymptotic central limit theorem (see Lecture 21) implies that if $X_1, X_2, \dots$ are i.i.d. random variables with zero mean, unit variance, and finite absolute third moment, and $h : \mathbb{R} \to \mathbb{R}$ is a function with bounded third derivative, then $S_n = X_1 + \cdots + X_n$ satisfies for each n:

$$\left| \mathbb{E}\, h\Big(\frac{S_n}{\sqrt{n}}\Big) - \mathbb{E}\, h(Z) \right| \le \frac{C \|h'''\|_\infty \, \mathbb{E}|X_1|^3}{\sqrt{n}},$$

where $Z \sim N(0,1)$ and C is an absolute constant. Show that, in general, $\sqrt{n}$ cannot be replaced by any faster growing function of n.

Hint: Let X_n be Randemacher random variables, and make h vanish on the lattice $n^{-1/2}\mathbb{Z}$.

6. CLT has ups and downs, too

Let $X_1, X_2, \dots$ be i.i.d. random variables with zero mean and unit variance. Prove that, almost surely,

$$\limsup_n \frac{S_n}{\sqrt{n}} = +\infty; \quad \liminf_n \frac{S_n}{\sqrt{n}} = -\infty.$$

Why does this not contradict the CLT?

Hint: Fix any $M > 0$. Combining Fatou lemma and CLT, show that $\limsup_n \frac{S_n}{\sqrt{n}} > M$ holds with positive probability. Then automatically upgrade the probability to 1.

7. A non-example for CLT

Let $X_1, X_2, \dots$ be Rademacher random variables. Let $Y_1, Y_2, \dots$ be such that Y_k takes values $\pm k$ with probability $k^{-2}/2$ each and value 0 with probability $1 - k^{-2}$. Assume all these random variables are independent, and let $Z_k = X_k + Y_k$. Show that $S_n = Z_1 + \cdots + Z_n$ satisfies

$$\mathbb{E}\, S_n = 0 \text{ and } \operatorname{Var}\Big(\frac{S_n}{\sqrt{n}}\Big) \to 2 \quad \text{but} \quad \frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1).$$

Why does this example not contradict Lindeberg's CLT?

Hint: use Borel-Cantelli lemma to argue that $\frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \xrightarrow{a.s.} 0$.

8. Lyapunov's CLT

Check that Lindeberg's condition (ii) in CLT (see Lecture 24) can be replaced by the following condition:

(ii)′ there exists $\delta > 0$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}|X_{nk}|^{2+\delta} = 0.$$

Hint: check that Lyapunov's condition (ii)′ implies Lindeberg's condition (ii).

9. Spherical CLT

Let $X^{(n)} = (X_1^{(n)}, \dots, X_n^{(n)})$ be a random vector distributed uniformly on the Euclidean unit sphere in $\mathbb{R}^n$. Prove that the coordinates of $X^{(n)}$ are asymptotically normal, i.e. for any $k \in \mathbb{N}$ we have

$$\sqrt{n} X_k^{(n)} \xrightarrow{w} N(0,1) \quad \text{as } n \to \infty.$$

Hint: use rotation invariance to represent X as $X = Z/\|Z\|_2$ where $Z \sim N(0, I_n)$. Argue that $\|Z\|_2/\sqrt{n} \to 1$ a.s.

10. Poisson visits Gauss

Consider independent random variables $X_n \sim \text{Pois}(\lambda_n)$. Show that if $\lambda_n \to \infty$ then

$$\frac{X_n - \lambda_n}{\sqrt{\lambda_n}} \xrightarrow{w} N(0,1).$$

Hint: Compute the limit of the characteristic functions.

11. CLT for random sign sums

Let $X_1, X_2, \dots$ be independent Rademacher random variables. Let $a_1, a_2, \dots$ be a sequence of (nonrandom) numbers. Denote $m_n = \max_{k=1,\dots,n} a_k^2$ and $s_n = \sum_{k=1}^n a_k^2$. Show that if

$$m_n / s_n \to 0 \quad \text{as } n \to \infty,$$

then

$$\frac{1}{\sqrt{s_n}} \sum_{k=1}^{n} a_k X_k \xrightarrow{w} N(0,1).$$

Hint: Apply Lindeberg's CLT.

12. CLT with random number or terms

Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean zero and unit variance, and let $S_n = X_1 + \cdots + X_n$. Let N_n be a sequence of nonnegative integer-valued random variables and a_n be a (nonrandom) sequence of nonnegative integers such that $a_n \to \infty$ and $N_n/a_n \to 1$ in probability. Show that

$$\frac{S_{N_n}}{\sqrt{a_n}} \to N(0,1)$$

weakly.

(Hint: use Kolmogorov's maximal inequality to conclude that if $Y_n = S_{N_n}/\sqrt{a_n}$ and $Z_n = S_{a_n}/\sqrt{a_n}$, then $Y_n - Z_n \to 0$ in probability.)