HOMEWORK 6
PROBABILITY: A GRADUATE COURSE

1. Prokhorov's theorem

Let (X_n) be a sequence of random variables. Prove that the following two statements are equivalent:

(i) (X_n) is tight;
(ii) any subsequence of (X_n) has a weakly convergent subsequence.

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2. The most predictable and most unpredictable random variables

Fix any n-point set $B \subset \mathbb{R}$, and consider all random variables X that take values in B. Let $H(X)$ denote the entropy of X.

(a) Show that $0 \le H(X) \le \log_2 n$.
(b) Show that $H(X) = \log_2 n$ if and only if X is uniformly distributed on B.
(c) Show that $H(X) = 0$ if and only if X is constant, i.e. $\mathbb{P}\{X = b\} = 1$ for some $b \in B$.

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3. Kolmogorov's two series theorem is irreversible

Find a sequence of independent mean zero random variables (X_n) for which $\sum_{n=1}^{\infty} X_n$ converges almost surely, yet $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) = \infty$.

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4. Levy's random series theorem

Let (X_n) be a sequence of independent random variables. Prove that the series $\sum_{n=1}^{\infty} X_n$ converges in probability if and only if it converges almost surely.

Hint: modify the proof of Kolmogorov's two series theorem. Do not use Chebyshev's inequality, but do use Etemadi's maximal inequality.

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5. Does Helly's selection theorem yield convergence everywhere?

Our version of Helly's selection theorem guarantees the convergence of F_{n_k} to the limit F at all points of continuity x of F.

(i) Show by example that convergence may not hold for all $x \in \mathbb{R}$.

(ii) Show that if we drop the right continuity requirement for F, we can achieve convergence for all $x \in \mathbb{R}$.

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6. When are normal random variables tight?

Let $X_n \sim N(\mu_n, \sigma_n^2)$. Find necessary and sufficient conditions, in terms of μ_n and σ_n, for tightness of the sequence (X_n).

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7. Scheffé's lemma

(a) (For continuous distributions) Prove that if the probability density functions of X_n converge to the probability density function of X pointwise, then X_n converges to X weakly.

(b) (For discrete distributions) Prove that if the probability mass functions of X_n converge to the probability mass function of X pointwise, then X_n converges to X weakly.

(c) (No converse) In general, weak convergence does not imply pointwise convergence of probability density functions. Find an example of random variables X_n with densities f_n so that X_n converge weakly to $X \sim \text{Unif}[0,1]$ but $f_n(x)$ does not converge to 1 for any $x \in [0,1]$.

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8. Extreme values

Let X_i be i.i.d. random variables each having exponential distribution with mean 1, i.e. $\mathbb{P}\{X_i > x\} = e^{-x}$ for all $x \in \mathbb{R}$. Consider

$$M_n := \max_{i \le n} X_i.$$

Show that $M_n - \log n$ converges weakly to the random variable Y with the standard Gubmel distribution, i.e. the random variable with cdf $F(x) = \exp(-e^{-x})$.

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9. Convergence to a constant

Let X_n be random variables and c be a constant. Prove that weak convergence of X_n to c is equivalent to convergence of X_n to c in probability.

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10. Convergence together

Consider the following statement:

$$\text{if } X_n \to X \text{ weakly and } Y_n \to Y \text{ weakly then } X_n + Y_n \to X + Y \text{ weakly.} \qquad (1)$$

(a) Find an example showing that (1) is false in general.

(b) Prove that if Y is a constant, then (1) is true.

(c) Prove that if X_n and Y_n are independent, and X and Y are independent, then (1) is true.