Homework 7
Probability: a Graduate Course

1. Increasing functions are positively correlated

Let X be a random variable and $f, g : \mathbb{R} \to \mathbb{R}$ be nondecreasing functions. Prove that random variables $f(X)$ and $g(X)$ are non-negatively correlated. Feel free to add any reasonable integrability assumptions.

Hint: Let Y be an independent copy of X. Note that the random variables $f(X) - f(Y)$ and $g(X) - g(Y)$ always have the same sign. Take expectation of their product.

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2. Sums of independent Poissons

Let $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$ be independent. Use characteristic functions to check that $X + Y \sim \text{Pois}(\lambda + \mu)$.

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3. The rate of convergence in CLT

The Wasserstein form of CLT (p. 136 of the notes) states that if $X_1, X_2, \ldots$ are i.i.d. random variables with zero mean, unit variance, and finite absolute third moment, then $S_n = X_1 + \cdots + X_n$ satisfies

$$W_1\Big(\frac{S_n}{\sqrt{n}}, N(0,1)\Big) = O\Big(\frac{1}{\sqrt{n}}\Big).$$

Find an example that shows that the rate $O(1/\sqrt{n})$ is optimal.

Hint: Let X_n be Randemacher random variables, and let $h(x)$ denote the distance from $x \in \mathbb{R}$ to the lattice $n^{-1/2}\mathbb{Z}$. Smooth h a bit to make $\|h'\|_\infty = O(1)$.

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4. CLT has ups and downs, too

Let $X_1, X_2, \ldots$ be i.i.d. random variables with zero mean and unit variance. Prove that

$$\limsup_n S_n/\sqrt{n} = +\infty, \quad \liminf_n S_n/\sqrt{n} = -\infty.$$

Why does this not contradict the CLT?

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5. A non-example for CLT

Let $X_1, X_2, \dots$ be Rademacher random variables. Let $Y_1, Y_2, \dots$ be such that Y_k takes values $\pm k$ with probability $k^{-2}/2$ each and value 0 with probability $1 - k^{-2}$. Assume all these random variables are independent, and let $Z_k = X_k + Y_k$. Show that $S_n = Z_1 + \cdots + Z_n$ satisfies

$$\mathbb{E}\, S_n = 0 \text{ and } \operatorname{Var}\left(\frac{S_n}{\sqrt{n}}\right) \to 2 \quad \text{but} \quad \frac{S_n}{\sqrt{n}} \xrightarrow{w} N(0,1).$$

Why does this example not contradict Lindeberg's CLT?

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6. Lyapunov's CLT

Check that assumption (ii) in Lindeberg's CLT (p.128 of the notes) can be replaced by the following assumption:

(ii)$'$ there exists $\delta > 0$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}|X_{nk}|^{2+\delta} = 0.$$

Hint: check that Lyapunov's condition (ii)$'$ implies Lindeberg's condition (ii).

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7. Spherical CLT

Let $X^{(n)} = (X_1^{(n)}, \dots, X_n^{(n)})$ be a random vector distributed uniformly on the Euclidean unit sphere in $\mathbb{R}^n$. Prove that the coordinates of $X^{(n)}$ are asymptotically normal, i.e. for any $k \in \mathbb{N}$ we have

$$\sqrt{n} X_k^{(n)} \xrightarrow{w} N(0,1) \quad \text{as } n \to \infty.$$

Hint: use rotation invariance to represent X as $X = Z/\|Z\|_2$ where $Z \sim N(0, I_n)$. Argue that $\|Z\|_2 / \sqrt{n} \to 1$ a.s.

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8. Poisson visits Gauss

Consider independent random variables $X_n \sim \text{Pois}(\lambda_n)$. Show that if $\lambda_n \to \infty$ then

$$\frac{X_n - \lambda_n}{\sqrt{\lambda_n}} \xrightarrow{w} N(0,1).$$

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9. CLT for random sign sums

Let $X_1, X_2, \dots$ be independent Rademacher random variables. Let $a_1, a_2, \dots$ be a sequence of (nonrandom) numbers. Denote $m_n = \max_{k=1,\dots,n} a_k^2$ and $s_n = \sum_{k=1}^n a_k^2$. Show that if

$$m_n/s_n \to 0 \quad \text{as } n \to \infty,$$

then

$$\frac{1}{\sqrt{s_n}} \sum_{k=1}^n a_k X_k \xrightarrow{w} N(0,1).$$

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10. CLT with random number or terms

Let $X_1, X_2, \dots$ be i.i.d. random variables with mean zero and unit variance, and let $S_n = X_1 + \cdots + X_n$. Let N_n be a sequence of nonnegative integer-valued random variables and a_n be a (nonrandom) sequence of nonnegative integers such that $a_n \to \infty$ and $N_n/a_n \to 1$ in probability. Show that

$$S_{N_n}/\sqrt{a_n} \to N(0,1)$$

weakly.

(Hint: use Kolmogorov's maximal inequality to conclude that if $Y_n = S_{N_n}/\sqrt{a_n}$ and $Z_n = S_{a_n}/\sqrt{a_n}$, then $Y_n - Z_n \to 0$ in probability.)