Homework 8
Probability: a Graduate Course

1. A simple random walk

In all problems of this homework, we denote by S_n a simple random walk on $\mathbb{Z}$ starting at the origin. Prove that there exists an absolute constant $c > 0$ such that

$$c\sqrt{n} \le \mathbb{E}|S_n| \le \sqrt{n} \quad \text{for any } n \in \mathbb{N}.$$

2. Two simple questions about martingales

True or false? Prove or give a counterexample to each of the following statements.

(a) A sequence of integrable random variables (X_n) is a martingale (with respect to some filtration) if and only if

$$\mathbb{E}\left[X_{n+1}|X_1, \ldots, X_n\right] = X_n \quad \text{for any } n \in \mathbb{N}.$$

(b) Any sequence of random variables (X_n) satisfying

$$\mathbb{E}\, X_n = 0 \quad \text{for any } n \in \mathbb{N}$$

is a martingale with respect to some filtration.

3. A cubic martingale

In Lecture 31, we used the compensator method to show that $S_n^3 - \sum_{k=0}^{n-1} S_k$ is a martingale. Deduce from this fact that

$$S_n^3 - 3S_n$$

is a martingale.

Hint: In order to replace each term S_k in the sum by S_n, prove and use the following fact: if (R_n) is a martingale, $A_n := \sum_{k=0}^{n-1} R_k$ and $B_n := nR_n$, then $\mathbb{E}[A_{n+1} - A_n|\mathcal{F}_n] = \mathbb{E}[B_{n+1} - B_n|\mathcal{F}_n]$.

4. A cubic martingale, simplified?

Prove or disprove the following statement. There exists a deterministic sequence of real numbers (c_n) such that

$$S_n^3 - c_n$$

is a martingale.

5. A quartic martingale

Let S_n be a simple random walk on $\mathbb{Z}$, starting at the origin. Use the compensator method to find f_n that makes

$$S_n^4 - f_n(S_1, \ldots, S_{n-1})$$

is a martingale.

6. Optional switching theorem

Let (X_n) and (Y_n) be two martingales with respect to the same filtration $(\mathcal{F}_n)$. Let T be a stopping time. Assume that

$$X_T = Y_T \text{ a.s.}$$

Show that

$$Z_n := \begin{cases} X_n, & n < T \\ Y_n, & n \geq T \end{cases}$$

is a martingale.

7. Fisher-Wright model

In Lecture 33, we studied the Fisher-Wright model in population genetics. We showed that if the population size grows sublinearly ($n_t/t \to 0$) then the population eventually becomes homogeneous a.s. (only one type survives).

Suppose now that the the population size grows superlinearly:

$$n_t/t^{1+\varepsilon} \to \infty \quad \text{as } t \to \infty$$

for some $\varepsilon > 0$. Show that the population remains heterogeneous: at any given time t, the expected fraction of individuals of type A is at least $c(p_0, \varepsilon) > 0$, and same for type B.

Hint: following the analysis in Lecture 33, give a lower bound on $\mathbb{E}\, h_t$.

8. Moran model

Consider the Polya urn model with removal. Start with a population with n individuals, $p_0 n$ of which are of type A and $(1 - p_0)n$ are of type B. At each step,

- one person is randomly chosen to reproduce (producing one offspring of the same type);
- one person is randomly chosen to die.

Formalize this model and analyze its homogenization, just like we did in Lecture 33 for the Fisher-Wright model.

9. Criterion of uniform integrability

In Lecture 34, we proved that a sequence of random variables (X_n) is uniformly integrable if and only if both of the following conditions hold:

(a) $\sup_n \mathbb{E}|X_n| < \infty$;
(b) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any event satisfying $\mathbb{P}(F) < \delta$, we have $\sup_n \mathbb{E}\left[|X_n|\mathbf{1}_F\right] < \varepsilon$.

Does (b) imply (a)? Prove or disprove.