Homework 8
Probability: a Graduate Course

1. Conditioning on the sum

Let X and Y be independent and identically distributed random variables with finite means. Show that
$$\mathbb{E}[X|X+Y] = \frac{X+Y}{2}.$$

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2. Conditional Jensen's inequality

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, $\mathcal{F} \subset \Sigma$ be a sigma-algebra, $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function, and X be a random variable satisfying $\mathbb{E}|X| < \infty$ and $\mathbb{E}|\varphi(X)| < \infty$. Prove that
$$\varphi\left(\mathbb{E}[X|\mathcal{F}]\right) \leq \mathbb{E}\left[\varphi(X)|\mathcal{F}\right].$$

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3. Gauss meets Rademacher

Let $X \sim N(0,1)$ and $Y = |X|$. Show that $X|Y$ has the same distribution as rY where r is an independent Rademacher random variable.

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4. CLT for random polynomials

Let X_i be independent standard normal random variables.

(a) Prove a limit theorem for $S_n = \sum_{i=1}^{n-1} X_i X_{i+1}$.

(b) Prove a limit theorem for $T_n = \sum_{i,j=1}^{n} X_i X_j$.

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5. The law of total variance

Define the conditional variance of X on the sigma-algebra $\mathcal{F}$ as
$$\mathrm{Var}(X|\mathcal{F}) = \mathbb{E}[X^2|\mathcal{F}] - \left(\mathbb{E}[X|\mathcal{F}]\right)^2.$$

Check that
$$\mathrm{Var}(X) = \mathbb{E}\left[\mathrm{Var}(X|\mathcal{F})\right] + \mathrm{Var}\left(\mathbb{E}[X|\mathcal{F}]\right).$$

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6. Conditional expectation is a contraction

Check that conditional expectation is a contraction in L^p. Specifically, let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, let $\mathcal{F} \subset \Sigma$ be a sigma-algebra, and let $p \in (1, \infty)$. For a random variable $X \in L^p = L^p(\Omega, \Sigma, \mathbb{P})$, denote $f(X) = \mathbb{E}[X|\mathcal{F}]$. Check that

$$\|f(X) - f(Y)\|_{L^p} \leq \|X - Y\|_{L^p} \quad \text{for any } X, Y \in L^p.$$

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7. Memoryless distributions

Show that the only absolutely continuous distribution that is memoryless is exponential. (Make this statement precise.)

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8. Conditioning on truncation

Let $X \sim \mathrm{Exp}(\lambda)$. Find $\mathbb{E}\left[X | \max(X, t)\right]$ for any fixed number $t > 0$.

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9. Conditional Cauchy-Schwarz inequality

Show that

$$(\mathbb{E}[XY|\mathcal{F}])^2 \leq \mathbb{E}[X^2|\mathcal{F}] \cdot \mathbb{E}[Y^2|\mathcal{F}]$$

almost surely.

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10. Conditioning reduces second moment

Let $Y = \mathbb{E}[X|\mathcal{F}]$. Show that if $\mathbb{E}[Y^2] = \mathbb{E}[X^2]$ then $X = Y$ a.s.
