FOURIER ANALYSIS BACKGROUND

Def The Fourier transform of an integrable function $f: \mathbb{R} \to \mathbb{R}$ is

$$\hat{f}(t) := \int_{-\infty}^{\infty} e^{-i2\pi t x} f(x)\,dx, \quad t \in \mathbb{R}.$$

Prop If f is integrable, $\hat{f}$ is bounded and continuous.

& converging to 0 at ∞ [Riemann-Lebesgue lemma] } ← WON'T BE NEEDED

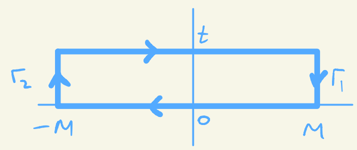
- $|e^{-i2\pi t x}| = 1 \Rightarrow |\hat{f}(t)| \leq \int_{-\infty}^{\infty} |f(x)|\,dx$
- $\hat{f}(t+h) = \int_{-\infty}^{\infty} \underbrace{e^{-i2\pi(t+h)x}}_{\to\, e^{-i2\pi t x}} f(x)\,dx \longrightarrow \hat{f}(t)$ by D.C.T (dominated by f)

(a) (Gaussian): $f(x) = e^{-\pi x^2} \Rightarrow \hat{f}(t) = e^{-\pi t^2}$

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-i2\pi t x} e^{-\pi x^2}\,dx = e^{-\pi t^2} \int_{-\infty}^{\infty} e^{-\pi(x+it)^2}\,dx \quad \text{(Complete the square)}$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx$$

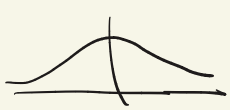
(by contour integration: e^{-z^2} is entire $\Rightarrow \oint_{\Gamma} e^{-z^2}\,dz = 0$. But $\int_{\Gamma_1, \Gamma_2} e^{-z^2}\,dz \xrightarrow{M\to\infty} 0$ (check!))

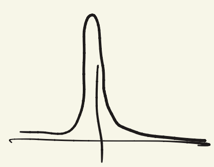


$= 1$ (Gaussian integral)

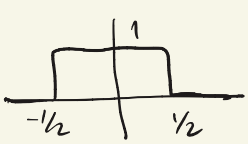
(b) (Scaling) $\widehat{f(\delta x)}(t) = \frac{1}{\delta} \hat{f}\left(\frac{x}{\delta}\right)$ (by change of var's) $\Rightarrow$

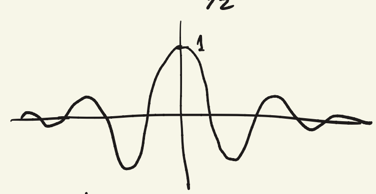
$$f(x) = e^{-\pi \delta^2 x^2} \Rightarrow \hat{f}(t) = \frac{1}{\delta} e^{-\pi x^2/\delta^2}$$



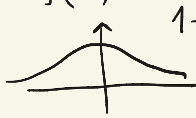


(c) $f(x) = \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}(x) \Rightarrow \hat{f}(t) = \int_{-1/2}^{1/2} e^{-i2\pi t x}\,dx \overset{y = -i2\pi t x}{=\!=} \frac{\sin(\pi t)}{\pi t} = \mathrm{sinc}(\pi t)$





(d) $f(x) = e^{-2|x|} \Rightarrow \hat{f}(t) = \frac{1}{1+(\pi t)^2}$



SMOOTHNESS & DECAY

- F.T. of the derivative: $\boxed{\widehat{f'}(t) = i2\pi t \hat{f}(t)}$ as long as f' is integrable

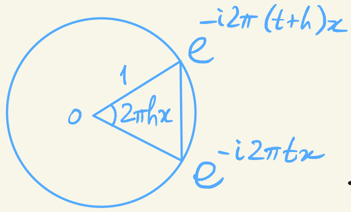
$$\int_{-\infty}^{\infty} f'(x) e^{-i2\pi t x}\,dx \overset{\text{by}}{\underset{\text{parts}}{=}} -\int_{-\infty}^{\infty} f(x) \frac{d}{dx}\left(e^{-i2\pi t x}\right) dx = i2\pi t \int_{-\infty}^{\infty} e^{-i2\pi t x} f(x)\,dx$$

- Derivative of the F.T: $\boxed{\frac{d}{dt}\hat{f}(t) = -i2\pi\, \widehat{xf(x)}(t)}$ as long as $xf(x)$ is integrable

$$\int_{-\infty}^{\infty} \underbrace{\frac{d}{dt}\left(e^{-i2\pi t x}\right)} f(x)\,dx = -i2\pi \int_{-\infty}^{\infty} e^{-i2\pi t x} x f(x)\,dx$$

Justification: $\lim_{h\to 0} \dfrac{e^{-i2\pi(t+h)x} - e^{-i2\pi t x}}{h}$. $|\text{Numerator}| \le |2\pi h x| \Rightarrow |\text{Ratio}| \le |2\pi x|$

$\Rightarrow |\text{Ratio} \cdot f(x)| \le 2\pi x f(x)$ integrable $\Rightarrow$ DCT applies.



- Induction $\Rightarrow$

(*) $\widehat{f^{(p)}}(t) = (i2\pi t)^p \hat{f}(t)$ as long as $f^{(p)}$ is integrable.

(**) $\hat{f}^{(q)}(t) = (-i2\pi)^q \widehat{x^q f(x)}(t)$ as long as $x^q f(x)$ is integrable.

- Since F.T. of an integrable function is always bounded, this yields:

Prop (smoothness & decay) $\forall\, p, q \in \{0, 1, 2, \dots\}$:

(a) $f^{(p)}(x)$ is integrable $\Rightarrow$ $t^p \hat{f}(t)$ is bounded.

(b) $x^q f(x)$ is integrable $\Rightarrow$ $\hat{f}^{(q)}(x)$ is bounded.

i.e. (a) f is smooth $\Rightarrow$ $\hat{f}$ decays fast; (b) f decays fast $\Rightarrow$ $\hat{f}$ is smooth

- Revisit the examples above.

• Combining the 2 heurics: f is both smooth & decays fast $\Rightarrow \hat{f}$ is, too. Formally:

Def Schwartz space $S := \{ f : \mathbb{R} \to \mathbb{R} : \; x^p f^{(q)}(x) \text{ is bounded } \forall p, q = 0, 1, 2, \dots \}$

i.e. f & all derivatives decay fast.

Prop $f \in S \Rightarrow \hat{f} \in S$

$g(x) := x^q f(x) \in S \Rightarrow g^{(p)}(x)$ is integrable $\xRightarrow{\text{Prop (a)}} t^p \hat{g}(t)$ is bounded.

But $\hat{g}(t) = \widehat{x^q f(x)}(t) \overset{(*)}{=\!=} (-i2\pi)^{-q} \hat{f}^{(q)}(t) \Rightarrow t^p \hat{f}^{(q)}(t)$ is bounded

THM (Fourier inversion formula) If $f, \hat{f} \in$ Schwartz space, then

$$f(x) = \int_{-\infty}^{\infty} e^{i2\pi t x} \hat{f}(t)\, dt.$$

- WLOG, $x=0$ (by translation).

$$f(0) \overset{?}{=} \int_{-\infty}^{\infty} \hat{f}(t)\, dt$$

- Physics "proof":

$$\overset{\text{def, Fubini}}{=} \int_{-\infty}^{\infty} f(x) \underbrace{\left(\int_{-\infty}^{\infty} e^{-i2\pi t x} dt \right)}_{= \text{Dirac } \delta_0(x)} dx = f(0), \quad \text{Q.E.D.}$$

- Rigorous proof: "Gaussian mollification": insert $e^{-\pi \delta^2 t^2} =: G_\delta(t)$ here.

So, instead of $\int_{-\infty}^{\infty} \hat{f}(t)\, dt$, compute the "mollified" version:

$$\int_{-\infty}^{\infty} \hat{f}(t) \underbrace{G_\delta(t)}_{\downarrow \text{pointwise } 1} dt = \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} e^{-i2\pi t x} G_\delta(t)\, dt \right) dx$$

- $\delta \downarrow 0$: $\underbrace{\qquad 1 \qquad}$ $\wedge |\hat{f}(t)|$ integrable

$\xrightarrow{\text{DCT}} \int_{-\infty}^{\infty} \hat{f}(t)\, dt$

$$= \int_{-\infty}^{\infty} f(x) \underbrace{\hat{G}_\delta(x)}_{} dx$$

$\overset{\text{Ex. above}}{=} \frac{1}{\delta} e^{-\pi x^2/\delta^2} = \text{pdf of } X_\delta \sim N\left(0, \frac{\delta^2}{2\pi}\right) \xrightarrow{d} 0$

$\left(\text{since } X_\delta \sim \frac{\delta}{\sqrt{2\pi}} N(0,1)\right)$

$$= \mathbb{E} f(X_\delta) \longrightarrow f(0) \text{ by Portmanteau lemma (iv)}$$

$$\Rightarrow \int_{-\infty}^{\infty} \hat{f}(t)\, dt = f(0), \quad \text{Q.E.D}$$

Remark | Fourier inversion formula is valid more generally — if both f and $\hat{f}$ are integrable (see below).

The L^2 theory (optional)

THM (Plancherel identity) $\forall\, f,g \in S$:

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} \hat{f}(t)\overline{\hat{g}(t)}\,dt \qquad \text{i.e. } \langle f,g\rangle_{L^2} = \langle \hat{f},\hat{g}\rangle_{L^2}$$

In particular, for $g=f$ we get

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(t)|^2\,dt \qquad \text{i.e. } \|f\|_{L^2} = \|\hat{f}\|_{L^2}$$

Fourier inversion $\Rightarrow$ $g(x) = \int_{-\infty}^{\infty} e^{i2\pi tx}\hat{g}(t)\,dt$. Substitute $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} e^{-i2\pi tx}\,\overline{\hat{g}(t)}\,dt\right)dx \overset{\text{Fubini}}{=\!=} \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-i2\pi tx} f(x)\,dx\right)\overline{\hat{g}(t)}\,dt$$

$$= \int_{-\infty}^{\infty} \hat{f}(t)\overline{\hat{g}(t)}\,dt.$$

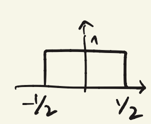
- $\Rightarrow$ F.T.: $f \mapsto \hat{f}$ is a linear isometry on $S \subset L^2$. But S is dense in L^2 $\Rightarrow$ F.T. extends (by continuity) to an isometry: $L^2 \to L^2$.
- $\Rightarrow$ Plancherel formula extends & holds in L^2.

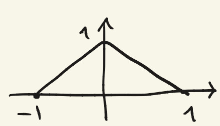
PROP (F.T. convolution $\to$ product) $\forall$ integrable f,g:

$$\widehat{f*g} = \hat{f}\cdot\hat{g} \quad \text{pointwise.}$$

Recall (Lec. 15): $f*g\,(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$

$$\widehat{f*g}(t) = \int_{-\infty}^{\infty} \underbrace{e^{-itx}}_{\substack{\| \\ e^{-ity}e^{-it(x-y)}}}\left(\int_{-\infty}^{\infty} f(y)g(x-y)\,dy\right)dx \underset{\text{Fubini-Tonelli}}{=} \int_{-\infty}^{\infty} e^{-ity} f(y)\left(\underbrace{\int_{-\infty}^{\infty} e^{-it(x-y)} g(x-y)\,dx}_{\substack{\| \, z=x-y \\ \int_{-\infty}^{\infty} e^{-itz}g(z)\,dz}}\right)dy = \hat{f}(t)\cdot\hat{g}(t).$$

Ex $f =$ 

$f*f =$ 

$\widehat{f*f}(t) = \hat{f}(t)^2 = \left(\frac{\mathrm{sinc}(\pi x)}{\pi x}\right)^2 =$