FOURIER ANALYTIC PROOF OF CLT

Heuristic:

- By Portmanteau lemma, $X_n \xrightarrow{d} X \iff \mathbb{E}h(X_n) \to \mathbb{E}h(X)$ $\forall$ "nice" h $(*)$
- Fourier analysis $\Rightarrow$ enough to consider functions of the form $h(x) = e^{itx}$, i.e.

$$X_n \xrightarrow{d} X \iff \mathbb{E}e^{itX_n} \to \mathbb{E}e^{itX} \quad \forall t \in \mathbb{R}$$

- CLT for mean 0, var 1:

$$\mathbb{E}e^{it(X_1+\dots+X_n)/\sqrt{n}} = \prod_{i=1}^{n} \mathbb{E}e^{itX_i/\sqrt{n}} = \left(\mathbb{E}e^{itX/\sqrt{n}}\right)^n \quad =$$

- Taylor: $e^{itx} \approx 1 + itx + \frac{(itx)^2}{2}$. Substitute $x = X$, take $\mathbb{E} \Rightarrow$

$$\mathbb{E}e^{itX} \approx 1 - \frac{t^2}{2}$$

$$= \left(1 - \frac{t^2}{2n}\right)^n \to e^{-t^2/2} = \mathbb{E}e^{itZ} \quad \text{where } Z \sim N(0,1) \qquad \square.$$

- $\mathbb{E}e^{itX} = \int e^{itX} f(x)\,dx$ = Fourier transform of f.   FORMALLY

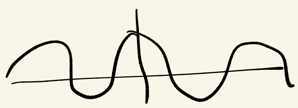
pdf

Back to probability:

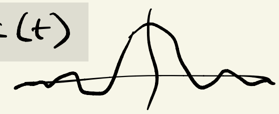
Def The characteristic function of a r.v. X is
$$\phi_X(t) = \phi(t) = \mathbb{E}e^{itX}, \qquad t \in \mathbb{R}$$

- Connection to Fourier transform:
  if X has density $f(x)$, $\phi(-2\pi t) = \int_{-\infty}^{\infty} e^{-i2\pi tx} f(x)\,dx = \hat{f}(t)$
  Generally, $\phi(-2\pi t) = \int_{-\infty}^{\infty} e^{-i2\pi tx}\, d\mathcal{L}_X(x)$ defines Fourier transform of a measure.

- Examples
  (a) $X \sim \text{Rademacher} \Rightarrow \phi(t) = \dfrac{e^{it}+e^{-it}}{2} = \boxed{\cos t}$



  (b) $X \sim \text{Unif}[-1,1] \Rightarrow \phi(t) = \int_{-1}^{1} e^{itx}\cdot\frac{1}{2}\,dx \underset{y=-itx}{=} \dfrac{\sin t}{t} = \boxed{\operatorname{sinc}(t)}$

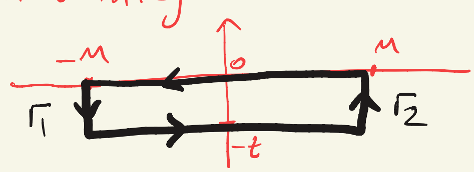


  (c) $X \sim N(0,1) \Rightarrow \phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx} e^{-x^2/2}\,dx = \boxed{e^{-t^2/2}}$ $(*)$

  (rescaling Ex. before)

  Direct proof: $\phi(t) = e^{-t^2/2}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-it)^2/2}\,dx}$ (complete the square)
  $$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1$$

  by contour integration



  $z \mapsto e^{-z^2/2}$ is entire $\Rightarrow \oint_\Gamma e^{-z^2/2}\,dz = 0$; $\int_{\Gamma_{1,2}} e^{-z^2/2}\,dz \to 0$ as $M \to \infty$

  (d) $X \sim \text{Cauchy}$: pdf $f = \dfrac{1}{\pi(1+x^2)} \Rightarrow \phi(t) = e^{-|t|}$ (rescaling an Ex. before)

Fact If X, Y are independent, $\phi_{X+Y} = \phi_X \cdot \phi_Y$

$\Big[\, e^{it(X+Y)} = e^{itX}\cdot e^{itY}$, take $\mathbb{E}$. $\Big]$

- Characteristic function ϕ_X determines the distribution of X uniquely. Moreover, stability holds:

THM (Lévy's continuity theorem)

$X_n \xrightarrow{d} X \iff \phi_{X_n} \to \phi_X$ pointwise.

$\Rightarrow$ In particular, uniqueness: $X \overset{d}{=} Y \iff \phi_X = \phi_Y$ everywhere.

Proof

($\Rightarrow$) $\forall$ fixed t, $x \mapsto e^{itx}$ is bdd, continuous.

$\Rightarrow$ by Portmanteau lemma, $\mathbb{E}e^{itX_n} \to \mathbb{E}e^{itX}$. $\square$

($\Leftarrow$) By P.L., it is enough to show that

$$\mathbb{E}h(X_n) \to \mathbb{E}h(X) \quad \forall h \in S.$$

Schwartz space

$$\Vert \quad \mathbb{E}\int_{-\infty}^{\infty} e^{i2\pi t X_n}\hat{h}(t)dt \quad \text{(Fourier inversion formula)}$$

$$= \int_{-\infty}^{\infty} \underbrace{\mathbb{E}[e^{i2\pi t X_n}]}_{\downarrow \ \mathbb{E}[e^{i2\pi tX}] \text{ by assumption}} \hat{h}(t)\,dt \quad (\text{Fubini-Tonelli}: \hat{h} \in S \text{ by Prop}\ldots \ \& \ |e^{i\cdots}| = 1)$$

$$\xrightarrow{DCT} \int_{-\infty}^{\infty} \mathbb{E}[e^{i2\pi tX}]\hat{h}(t)dt \quad (\text{by assumption}; \ |\phi_X(\cdot)| \le 1 \Rightarrow \text{dominated by } \hat{h} \in S)$$

$$= \mathbb{E}h(X) \quad (\text{repeating the first steps in reverse for } X_n \mapsto X) \quad \square$$

Quick application of uniqueness

① Sum of normals = normal: if $X_i \sim N(0,\sigma_i^2)$ are indep then $\sum_i X_i \sim N(0,\sigma^2)$, $\sigma^2 = \sum_i \sigma_i^2$

$\ulcorner \phi_{X_i}(t) = e^{-t^2\sigma_i^2/2}$ (ex. before) $\Rightarrow \phi_{\Sigma X_i}(t) = \prod_i \phi_i(t) = \prod_i e^{-t^2\sigma_i^2/2} = e^{-t^2\sigma^2/2} = \phi_Z(t)$ for $Z \sim N(0,\sigma^2)$ $\lrcorner$

② Sum of Cauchy is Cauchy: If $X_i \sim$ Cauchy are indep then $\frac{1}{n}\sum_{i=1}^n X_i \sim$ Cauchy

$\ulcorner \phi_{X_i}(t) = e^{-|t|}$ (Ex. before) $\Rightarrow$ repeat the argument in ①. $\lrcorner$

Lem (Taylor expansion of the ch. function)

Let X be a r.v. with $\mathbb{E}X=0$, $\mathbb{E}X^2=\sigma^2<\infty$. Then

$$\phi_X(t) = 1 - \frac{t^2\sigma^2}{2} + \bar{o}(t^2) \quad \text{as } t\to 0.$$

Proof Use Taylor formula from the first proof of classical CLT for e^{itx}, noting that e^{itx} & all of its derivatives are $|\cdot|\le 1$:

$$\left| e^{ix} - 1 - ix + \frac{x^2}{2} \right| \le 4\min\left(|x|^3, x^2\right) \quad \forall x\in\mathbb{R}$$

• Substitute $x=tX$, take $\mathbb{E}$ and push it inside $|\cdot|$ by Jensen $\Rightarrow$

$$\left| \mathbb{E}e^{itX} - 1 - it\mathbb{E}X + \frac{t^2\mathbb{E}X^2}{2} \right| \le 4\,\mathbb{E}\min\left(t^3|X|^3, t^2|X|^2\right)$$

$$\| \qquad = 4t^2\cdot \mathbb{E}\,\underbrace{\min\left(t|X|^3, X^2\right)}$$

$$\left| \phi_X(t) - 1 + \frac{t^2\sigma^2}{2} \right|$$

this r.v. is $\le X^2$ and $\to 0$ everywhere as $t\to 0$

DCT $\Rightarrow$ $\mathbb{E}\min(\cdot)\to 0$ $\square$.

Examples: (a) $X\sim N(0,\sigma^2) \Rightarrow \phi_X(t) = e^{-t^2\sigma^2/2} \approx 1 - \frac{t^2\sigma^2}{2}$.

(b) $X\sim$ Cauchy $\quad \phi_X(t) = e^{-|t|}$. The cusp at 0 reflects $\nexists$ mean.

Let's reprove

Classical CLT Let $X_1, X_2, \dots$ be iid. r.v's with mean μ and variance σ^2.
Then $S_n = X_1 + \dots + X_n$ satisfies

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{w} Z \sim N(0,1)$$

Proof WLOG $\mu=0, \sigma=1$ $\Rightarrow$ by Lem above,

$$\mathbb{E}e^{i\tau X_1} = 1 - \frac{\tau^2}{2} + \bar{o}(\tau^2) = 1 - \frac{\tau^2}{2}(1+\bar{o}(1)) \quad \text{as } \tau \to 0 \quad (*)$$

Fix $\forall t \in \mathbb{R}$.

$$\mathbb{E}e^{itS_n/\sqrt{n}} = \mathbb{E}\prod_{k=1}^{n} e^{itX_k/\sqrt{n}} \overset{\text{iid}}{=\!=} \left(\mathbb{E}e^{itX_1/\sqrt{n}}\right)^n \underset{\tau := t/\sqrt{n}}{\overset{(*)}{=\!=}} \left[1 - \frac{t^2}{2n}(1+\bar{o}(1))\right]^n$$

$$\to e^{-t^2/2} \qquad \text{by Fact: } \boxed{z_n \to z \text{ in } \mathbb{C} \Rightarrow \left(1+\frac{z_n}{n}\right)^n \to e^z}$$

$$= \mathbb{E}e^{itZ} \quad \text{where } Z \sim N(0,1)$$

- So we showed $\phi_{S_n/\sqrt{n}} \to \phi_Z$ pointwise.

Lévy continuity thm $\Rightarrow$ $\frac{S_n}{\sqrt{n}} \xrightarrow{d} Z$. $\square$.

HW The identical distribution assumption is essential in the classical CLT. Find an example showing this.