Properties of Conditional Expectation

Rule of thumb: whatever properties expectation satisfies, conditional expectation satisfies a.s.:

Prop (a) (linearity) $\mathbb{E}[aX+bY\mid\mathcal{F}] = a\,\mathbb{E}[X\mid\mathcal{F}] + b\,\mathbb{E}[Y\mid\mathcal{F}]$ a.s.

(b) (Monotonicity) $X \le Y$ a.s. $\Rightarrow \mathbb{E}[X\mid\mathcal{F}] \le \mathbb{E}[Y\mid\mathcal{F}]$ a.s.

(c) (MCT) If $X_n \ge 0$, $X_n \uparrow X$ a.s., $\mathbb{E}X < \infty$, then $\mathbb{E}[X_n\mid\mathcal{F}] \uparrow \mathbb{E}[X\mid\mathcal{F}]$ a.s.

(d) (Fatou) If $X_n \ge 0$, $\mathbb{E}\liminf_n X_n < \infty$ then $\mathbb{E}[\liminf_n X_n\mid\mathcal{F}] \le \liminf_n \mathbb{E}[X_n\mid\mathcal{F}]$ a.s.

(e) (DCT) If $|X_n| \le Z$, $\mathbb{E}Z < \infty$, $X_n \to X$ a.s. then $\mathbb{E}[X_n\mid\mathcal{F}] \to \mathbb{E}[X\mid\mathcal{F}]$ a.s. (and in L^1)

by DCT

(f) (Jensen) If $\varphi:\mathbb{R}\to\mathbb{R}$ is convex, $\mathbb{E}|X|<\infty$, $\mathbb{E}|\varphi(X)|<\infty$, then $\varphi(\mathbb{E}[X\mid\mathcal{F}]) \le \mathbb{E}[\varphi(X)\mid\mathcal{F}]$ a.s.

Proof of (c) $R_n := X - X_n \downarrow 0$ a.s. It suffices to show that

$$E_n := \mathbb{E}[R_n\mid\mathcal{F}] \downarrow 0.$$

Monotonicity (b) $\Rightarrow E_n \downarrow$ a.s. and $E_n \ge 0$

$\Rightarrow E_n \downarrow Z$ a.s for some r.v. $Z \ge 0$.

$$\left.\begin{array}{l} \overset{\text{usual MCT}}{\Longrightarrow} \mathbb{E}E_n \longrightarrow \mathbb{E}Z \\ \text{(def. of cond. exp. w/ } \mathcal{F}=\Omega) \ \| \\ \mathbb{E}R_n \xrightarrow{\text{usual MCT}} 0 \end{array}\right\} \Rightarrow \mathbb{E}Z = 0 \overset{Z\ge 0}{\Longrightarrow} Z \overset{\text{a.s.}}{=} 0. \quad \square$$

Some properties specific to conditional expectation :

Prop

(a) (law of total expectation) $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X]$ a.s. ← trivial from def $(\mathcal{F} := \Omega)$

(b) (tower property) $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_1]|\mathcal{F}_2] = \begin{cases} \mathbb{E}[X|\mathcal{F}_1] & \text{if } \mathcal{F}_1 \subset \mathcal{F}_2 \\ \mathbb{E}[X|\mathcal{F}_2] & \text{if } \mathcal{F}_2 \subset \mathcal{F}_1 \end{cases}$ ← easy, too

(c) (independence) $X \perp\!\!\!\perp \mathcal{F} \Rightarrow \mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$ a.s. ← (trivial from def)

(c') (removing what's independent) If $\mathcal{F} \perp\!\!\!\perp \{\sigma(X), \mathcal{G}\}$ then

$$\mathbb{E}[X|\sigma(\mathcal{F}\cup\mathcal{G})] = \underbrace{\mathbb{E}[X|\mathcal{G}]}_{Z}$$

(d) (Factoring out what's known): X is $\mathcal{F}$-mble $\Rightarrow \mathbb{E}[XY|\mathcal{F}] = X \cdot \underbrace{\mathbb{E}[Y|\mathcal{F}]}_{= Z}$ a.s.

Proof

(c') Check def of conditional expectation

- Z is $\sigma(\mathcal{G}) \subset \sigma(\mathcal{F}\cup\mathcal{G})$-mble ✓
- $\forall\ E \in \sigma(\mathcal{F}\cup\mathcal{G})$: $\mathbb{E}[X \mathbb{1}_E] \overset{?}{=} \mathbb{E}[Z\mathbb{1}_E]$ (*)

① $\sigma(\mathcal{F}\cup\mathcal{G})$ is generated by sets of the form $E = F \cap G$, $F \in \mathcal{F}$, $G \in \mathcal{G}$.

For each such set, (*) holds:

$$\mathbb{E}[Z\mathbb{1}_E] = \mathbb{E}[\underbrace{\mathbb{E}[X|\mathcal{G}]\mathbb{1}_G}_{\text{independent}}\ \mathbb{1}_F] = \underbrace{\mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbb{1}_G]}_{\overset{\text{tower}}{=} \mathbb{E}[\mathbb{E}[X\mathbb{1}_G|\mathcal{G}]]} \cdot \mathbb{E}[\mathbb{1}_F] \overset{\text{indep}}{=} \mathbb{E}[X \underbrace{\mathbb{1}_F\mathbb{1}_G}_{= \mathbb{1}_E}]. \checkmark$$

② $\mathcal{P} := \{F\cap G : F \in \mathcal{F}, G\in\mathcal{G}\}$ is a π-system

$\mathcal{L} := \{E \in \sigma(\mathcal{F}\cup\mathcal{G})$ satisfying (*)$\}$ is a λ-system (by DCT)

$\mathcal{P} \subset \mathcal{L}$ by ① $\xRightarrow{\pi\text{-}\lambda \text{ thm}} \underset{\overset{\|①}{\sigma(\mathcal{F}\cup\mathcal{G})}}{\sigma(\mathcal{P})} \subset \mathcal{L}$. $\Rightarrow$ (*) holds $\forall\ E \in \sigma(\mathcal{F}\cup\mathcal{G})$.

(d) We need to check:

$$\mathbb{E}[XZ\mathbb{1}_F] = \mathbb{E}[XY\mathbb{1}_F] \quad \forall F \in \mathcal{F}.$$

1. for indicators $X = \mathbb{1}_E$, $E \in \mathcal{F}$

$$\mathbb{E}[Z\mathbb{1}_{E\cap F}] = \mathbb{E}[Y\mathbb{1}_{E\cap F}]$$

by def of $Z = \mathbb{E}[Y|\mathcal{F}]$, since $E\cap F \in \mathcal{F}$

2. for simple r.v.'s X: use linearity

3. for $X, Y \geq 0$: take simple r.v.'s $X_n \uparrow X$, use CMCT (Prop(c) p.148).

4. for general integrable X, Y: decompose $X = X^+ - X^-$, $Y = Y^+ - Y^-$. $\square$

Conditional expectation as a projection

- $L^2(\Omega,\Sigma,\mathbb{P}) = \{$ all r.v's X such that $\mathbb{E}X^2 < \infty\}$ Hilbert space:

  Inner product: $\langle X, Y\rangle := \mathbb{E}XY$ Norm: $\|X\| = \sqrt{\langle X,Y\rangle}$

- Fix $\mathcal{F} \subset \Sigma$.

  $L_{\mathcal{F}} := \{$ all r.v's $X \in L^2(\Omega,\Sigma,\mathbb{P})$ that are $\mathcal{F}$-measurable$\}$ $= L^2(\Omega,\mathcal{F},\mathbb{P})$

  ↑ closed linear subspace of $L^2(\Omega,\Sigma,\mathbb{P})$.

Prop (Projection) $P: X \mapsto \underbrace{\mathbb{E}[X|\mathcal{F}]}_{= Y}$ is the orthogonal projection in $L^2(\Omega,\Sigma,\mathbb{P})$ onto $L_{\mathcal{F}}$.

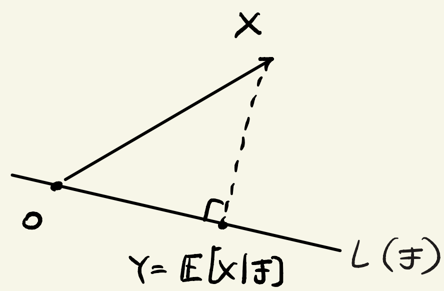
Proof P is a linear map in $L^2(\Omega,\Sigma,\mathbb{P})$ onto $L_{\mathcal{F}}$, ~~identity on $L_{\mathcal{F}}$~~

$\Rightarrow$ enough to check: $\quad Y \perp X - Y$

$$\Leftrightarrow \langle Y, X-Y\rangle = 0$$

$$\Leftrightarrow \mathbb{E}[Y(X-Y)] = 0$$

$$\Leftrightarrow \mathbb{E}[YX] = \mathbb{E}[Y^2]$$



L.T.E (Prop(a) p.149) $\Vert$

$$\mathbb{E}\big[\mathbb{E}[YX|\mathcal{F}]\big] = \mathbb{E}\big[Y\,\underbrace{\mathbb{E}[X|\mathcal{F}]}_{= Y}\big] = \mathbb{E}[Y^2]. \quad ☺ \quad \square$$

Y is $\mathcal{F}$-mble
conditional const (Prop(d) p.149)

Cor (contraction) $\quad Y := \mathbb{E}[X|\mathcal{F}] \Rightarrow \mathbb{E}Y^2 \le \mathbb{E}X^2$

Y

- Orthogonal proj outputs the nearest point in $L_{\mathcal{F}}$ $\Rightarrow$

Cor (Best predictor) $\quad \mathbb{E}[X|\mathcal{F}] = \underset{Y:\ \mathcal{F}\text{-mble}}{\operatorname{argmin}} \mathbb{E}(X-Y)^2$

HW: prove that $\mathbb{E}[\cdot|\mathcal{F}]$ is a contraction in L^p (p. 54 2019)

Conditioning on a random variable

Ex True/False? $E[X|aY+bZ] \stackrel{?}{=} aE[X|Y] + bE[X|Z]$

Def Let X, Y be r.v's. $\quad E[X|Y] := E[X|\sigma(Y)]$

Remarks ① By the Doob-Dynkin lemma (p.41),

$$E[X|Y] = h(Y) \quad \text{for some mble function } h.$$

② By the Cor. (prev. page),

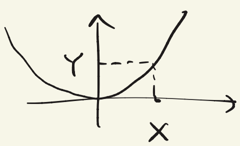
$E[X|Y]$ = best predictor of $E[X]$ constructed from Y

$$E[X|Y] = \underset{h:\mathbb{R}\to\mathbb{R} \text{ mble}}{\operatorname{argmin}} E(h(Y)-Y)^2$$

③ $E[X|Y]$ allows us to condition on events of meas. 0.

Ex 1 $X \geq 0$, $Y = X^2 \Rightarrow E[X|Y] = \sqrt{Y} = X$

[$\{X\in B\} = \{Y \in B^2\} \in \sigma(Y)$ $\forall$ Borel $B \Rightarrow X$ is $\sigma(Y)$-mble.]



Ex.2 $X \sim N(0,1)$, $Y = X^2 \Rightarrow E[X|Y] = 0$

[$\sigma(Y) = \{$origin-symmetric Borel sets in $\mathbb{R}\}$

$\Rightarrow \forall F \in \sigma(Y)$: $\quad E[X \mathbb{1}_F] = \int_F x \underbrace{f(x)}_{\text{pdf of } N(0,1)} dx = 0$ by symmetry.]

Ex.3 $X \sim \text{Unif}[0,n]$, $\quad Y := [X]$ (integer part)

$$\underset{[X]+\{X\}}{\underset{\parallel}{E[X|Y]}} = E[[X]|Y] + \underbrace{E[\{X\}|Y]}_{\text{indep. (Ex (c) p.43)}} = [X] + \underbrace{\{X\}}_{\text{Unif}[0,1]} = [X] + \frac{1}{2}.$$