Cover MARKOV CHAINS before martingales. Recurrence of 1D r. walk & absorbing states would immediately imply stopping times $<\infty$ a.s. in hitting 1, gamblers ruin, Fisher-Wright examples. we won't need to prove that.

MARTINGALES

Stochastic process: $X_1, X_2, \dots$

Def Let $(\Omega, \Sigma, \mathbb{P})$ be a prob. space.
Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subset \Sigma$ be a sequence of σ-algebras (a "filtration")
A sequence of integrable r.v's $(X_1, X_2, \dots)$ is called a *martingale* if $\forall n \in \mathbb{N}$:

(i) X_n is $\mathcal{F}_n$-measurable ("(X_n) is adapted to $(\mathcal{F}_n)$")

(ii) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ a.s. (no drift)

Remark: $\forall$ process $X_1, X_2 \dots$ can be made a martingale:
(i) choose $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$
(ii) subtract the drift — see later.

EXAMPLES:

(a) Simple random walk: $S_n = Z_1 + \dots + Z_n$, $Z_i \sim$ Rademacher iid

is a martingale w.r.to $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$

$$\mathbb{E}[\underbrace{S_{n+1}}_{=S_n + Z_{n+1}} \mid Z_1, \dots, Z_n] = \mathbb{E}[\underbrace{S_n}_{\text{mble}} \mid Z_1, \dots, Z_n] + \mathbb{E}[\underbrace{Z_{n+1}}_{\text{independent}} \mid Z_1, \dots, Z_n]$$

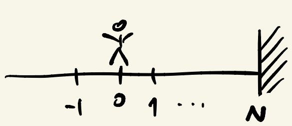
$$= S_n + \mathbb{E}[Z_{n+1}] = S_n.$$

(b) More generally, partial sums of $\forall$ mean zero r.v's

(c) Random walk with an absorbing wall:

$$X_0 := 0; \qquad X_{n+1} := \begin{cases} X_n + Z_{n+1} & \text{if } X_n < N \\ X_n & \text{if } X_n = N \end{cases}$$

iid Rademachers



(i) X_n is $\sigma(Z_1, \dots, Z_n)$-mble (X_n is determined by $Z_1, \dots, Z_n$)

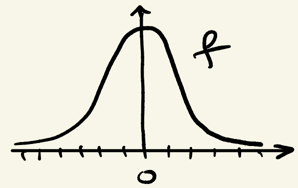
(ii) $\mathbb{E}[X_{n+1} \mid Z_1, \dots, Z_n] = X_n$ in either case.

- Convenient to express as $\boxed{X_n = S_{n \wedge T}}$

where $S_n = Z_1 + \dots + Z_n$ is a simple random walk,
$T = \min\{k : S_k = N\}$ is a "stopping time"

(d) More generally, a <u>slowed-down random walk</u>:

$$X_{n+1} := X_n + Z_{n+1} f(X_n) \qquad \forall f:\mathbb{R}\to\mathbb{R}, \text{ e.g.}$$



(e) <u>Quadratic martingale</u>:

If $S_n = Z_1 + \dots + Z_n$ where Z_i are iid mean 0 variance 1
then $S_n^2 - n$ is a martingale.

$$\mathbb{E}\big[\underset{\substack{\| \\ (S_n+Z_{n+1})^2}}{S_{n+1}^2} - (n+1) \,\big|\, Z_1,\dots,Z_n\big] = \mathbb{E}\big[\underbrace{S_n^2}_{\text{mble}} + 2\underbrace{S_n}_{\text{mble}} Z_{n+1} + \underbrace{Z_{n+1}^2}_{\text{indep}} \underbrace{- n - 1} \,\big|\, Z_1,\dots,Z_n\big]$$

$$= S_n^2 + 2S_n \underbrace{\mathbb{E}[Z_{n+1} \mid Z_1,\dots,Z_n]}_{\substack{\| \\ 0 \text{ (indep.)}}} + \underbrace{\mathbb{E}[Z_{n+1}^2]}_{\substack{\| \\ 1}} - n - 1 = S_n^2 - n.$$

(f) <u>Product martingale</u>

If $Z_1, Z_2, \dots$ be indep. r.v's with mean 1,
then $X_n := Z_1 \cdots Z_n$ is a martingale.

$$\mathbb{E}\big[\underset{\substack{\| \\ \underbrace{X_n}_{\text{mble}} Z_{n+1}}}{X_{n+1}} \,\big|\, Z_1,\dots,Z_n\big] = X_n \, \mathbb{E}\big[Y_{n+1} \,\big|\, Z_1,\dots,Z_n\big] = X_n \underbrace{\mathbb{E}[Z_{n+1}]}_{\substack{\| \\ 1}} = X_n$$

indep

(g) in particular, <u>St. Petersburg martingale</u>:

$$X_0 := 1, \quad X_{n+1} \mid X_n = \begin{cases} 2X_n, & \text{prob. } 1/2 \\ 0, & \text{prob. } 1/2 \end{cases} \qquad \text{(double the bet)}$$

$$\Rightarrow X_n = Z_1 \cdots Z_n \quad \text{where} \quad Z_k = \begin{cases} 2, & \text{prob. } 1/2 \\ 0, & \text{prob. } 1/2 \end{cases} \text{ iid.} \quad \mathbb{E}[Z_k] = 1.$$

(h) de Moivre's martingale

If $S_n := Z_1 + \dots + Z_n$ is a nonsymmetric r.walk: $Z_k = \begin{cases} 1, & \text{prob. } p \\ -1, & \text{prob. } q = 1-p \end{cases}$

then $X_n := \left(\frac{q}{p}\right)^{S_n}$ is a martingale

$$\Gamma \; X_n = \left(\frac{q}{p}\right)^{Z_1} \cdots \left(\frac{q}{p}\right)^{Z_n}, \qquad \mathbb{E}\left(\frac{q}{p}\right)^{Z_k} = \left(\frac{q}{p}\right)^{1} \cdot p + \left(\frac{q}{p}\right)^{-1} \cdot q = q + p = 1 \;\lrcorner$$

$\Rightarrow$ product martingale.

SKIP – counterintuitive.

(i) likelihood ratio test (recall from before):

Assume a r.v. X has density f or g.
Decide which one, based on an iid sample $X_1, \dots, X_n \sim X$.

Recall the solution: Likelihood Ratio $\Lambda_n := \dfrac{\prod_{i=1}^{n} f(x_i)}{\prod_{i=1}^{n} g(x_i)}$ $\begin{array}{ll} > 1 & \Rightarrow f \\ \leq 1 & \Rightarrow g \end{array}$

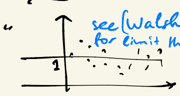
- If X has density g, then (Λ_n) is a martingale

$\Gamma \; \Lambda_n = Y_1 \cdots Y_n$ where $Y_i = \dfrac{f(x_i)}{g(x_i)}$ are iid r.v.'s

Remark: this is sufficient but NOT necessary: if $f \sim \text{Unif}[0,a]$, $g \sim \text{Unif}[0,a^2]$ and X has dens. f, then still $\mathbb{E}Y = \int \frac{f^2}{g} = 1$

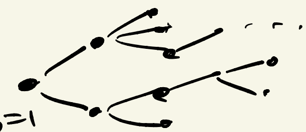
with $\mathbb{E}Y_i = \int_{-\infty}^{\infty} \frac{f(x)}{\cancel{g(x)}} \cdot \cancel{g(x)}\, dx = 1 \quad \Rightarrow$ product martingale. $\lrcorner$

- Hence, under the g-hypothesis, (Λ_n) does not "drift"



see [Walsh] for limit thm

(j) Branching processes



$X_0 = 1$

Galton-Watson process: model of the size of population

Assume each member of n'th generation gives birth independently to a random # of children, μ on average.

$X_n :=$ # individuals in n'th generation.

$$X_0 = 1; \quad X_{n+1} := \sum_{i=1}^{X_n} \xi_i^{(n)} \quad \text{where } \xi_i^{(n)} \geq 0 \text{ are iid with mean } \mu.$$

$$\mathbb{E}[X_{n+1} \mid X_n] = \sum_{i=1}^{X_n} \mathbb{E}\xi_i^{(n)} = \mu X_n$$

$\Rightarrow \dfrac{X_n}{\mu^n}$ is a martingale (divide by μ^n to check)

(k) Polya's urn

An urn initially contains N white and M black balls.
One ball is randomly drawn from an urn; its color is observed; it is returned to the urn, and an additional ball of the same color is added to the urn. Repeat.

(Motivation: rich get richer)

$X_n :=$ fraction of white balls after n'th step is a martingale

$\ulcorner$ If after n'th step the urn contains w white and b black balls, $\Rightarrow X_n = \frac{w}{w+b}$

$$X_{n+1} = \begin{cases} \frac{w+1}{w+b+1}, & \text{prob} = \frac{w}{w+b} \\ \frac{w}{w+b+1}, & \text{prob} = \frac{b}{w+b} \end{cases}$$

← white ball drawn

← black ball drawn

$$\mathbb{E}[X_{n+1} \mid w, b] = \frac{w+1}{w+b+1} \cdot \frac{w}{w+b} + \frac{w}{w+b+1} \cdot \frac{b}{w+b} = \frac{\cancel{(w+1+b)}\, w}{\cancel{(w+b+1)}(w+b)} = \frac{w}{w+b} = X_n \lrcorner$$

Intuition: each new ball is white with prob. = current white proportion.
$\Rightarrow$ $\mathbb{E}$# white balls doesn't change.

(l) Doob's martingale

Let X be a r.v. with $\mathbb{E}|X| < \infty$ and $(\mathcal{F}_n)$ be a filtration. Then

$$X_n := \mathbb{E}[X \mid \mathcal{F}_n]$$

is a martingale.

$$\ulcorner\ \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\big[\,\mathbb{E}[X \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n\big] = \mathbb{E}[X \mid \mathcal{F}_n] = X_n \lrcorner$$

SIMPLE PROPERTIES

Prop.1 $\mathbb{E}[X_n \mid \mathcal{F}_m] = X_{\underset{\uparrow \text{ min}}{n \wedge m}} \qquad \forall\, n, m \in \mathbb{N}$

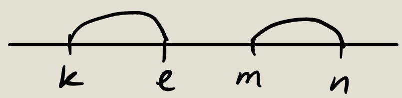
If $n \le m$ then X_n is $\mathcal{F}_n \subset \mathcal{F}_m$-mble $\Rightarrow \mathbb{E}[X_n \mid \mathcal{F}_m] = X_n$

If $n > m$ then $\mathbb{E}[X_n \mid \mathcal{F}_m] = \mathbb{E}\big[\underbrace{\mathbb{E}[X_n \mid \mathcal{F}_{n-1}]}_{= X_{n-1}} \mid \mathcal{F}_m\big] \overset{\text{repeat}}{=} \cdots = \mathbb{E}[X_m \mid \mathcal{F}_m] = X_m$

(L^2 mgles)

Prop 2 Martingale increments are mean zero & uncorrelated:

$$\forall\, k \le l \le m \le n : \quad \begin{cases} \mathbb{E}(X_l - X_k) = 0 \\ \mathbb{E}(X_l - X_k)(X_n - X_m) = 0 \end{cases}$$



- $X_k = \mathbb{E}[X_l \mid \mathcal{F}_k] \overset{\text{Tower}}{\Longrightarrow} \mathbb{E}X_k = \mathbb{E}X_l$

- $\mathbb{E}(X_l - X_k)(X_n - X_m) \overset{\text{Tower}}{=\!=} \mathbb{E}\big[\mathbb{E}[\underbrace{(X_l - X_k)}_{\mathcal{F}_m\text{-mble}}(X_n - X_m) \mid \mathcal{F}_m]\big]$

$$= \mathbb{E}\big[(X_l - X_k)\,\underbrace{\mathbb{E}[X_n - X_m \mid \mathcal{F}_m]}_{=\ \underbrace{\mathbb{E}[X_n \mid \mathcal{F}_m]}_{=\, X_m \text{ (Prop.1)}} - \underbrace{\mathbb{E}[X_m \mid \mathcal{F}_m]}_{=\, X_m} = 0}\big] = 0.$$

- Thus, $\forall$ martingale: $\boxed{X_n = X_0 + \sum_{k=1}^{n} (X_k - X_{k-1})}$

$\uparrow$ Sum of uncorrelated, mean 0 r.v's ("martingale differences") $\Rightarrow$ can prove WLLN.