SUBMARTINGALES, SUPERMARTINGALES

Def if, instead of $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$, we have

"$\geq$" $\Rightarrow$ submartingale

"$\leq$" $\Rightarrow$ supermartingale.

Examples: (a) Biased random walk: $S_n = Z_1 + \dots + Z_n$ where Z_i are indep, $\mathbb{E} Z_i \geq 0$

$$\left[\ \mathbb{E}[S_{n+1} \mid Z_1, \dots, Z_n] = S_n + \mathbb{E} Z_{n+1}\right] \geq S_n \ \right]$$

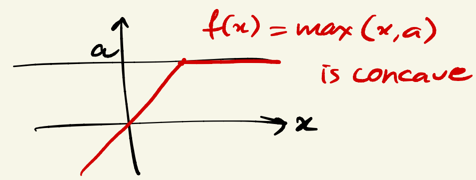
($S_{n+1} = S_n + Z_{n+1}$)

(b) if (X_n) is a martingale and $\varphi: \mathbb{R} \to \mathbb{R}$ is a convex function, then $(\varphi(X_n))$ is a submartingale.

$$\left[\ \text{Conditional Jensen (p. 148)} \Rightarrow \mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] \geq \varphi\left(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]\right) = \varphi(X_n) \ \right]$$

(c) In particular, if (X_n) is a martingale then:

- $|X_n|^p$ is a submartingale $\forall p \geq 1$;
- $\max(X_n, a)$ is a supermartingale $\forall a \in \mathbb{R}$



(d) If (X_n) is a martingale and (a_n) is a nondecreasing deterministic seq. $\Rightarrow$ $Y_n = X_n + h_n$ is a submartingale

$$\left[\ \mathbb{E}[Y_{n+1} - Y_n \mid \mathcal{F}_n] = \underbrace{\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]}_{=0} + \underbrace{h_{n+1} - h_n}_{\geq 0} \geq 0 \ \right]$$

($Y_{n+1} = X_{n+1} + h_{n+1}$, $Y_n = X_n + h_n$)

(e) (h_n) needs not be deterministic. It can be random but predictable:

Def A sequence of r.v's (H_n) is *predictable* if H_{n+1} is $\mathcal{F}_n$-measurable $\forall n$

i.e. can predict it "1 day ahead"

Example: H_n = our bet at time n (just before nth spin of the wheel) is based on what happened before (spins $1, \dots, n-1$)

THM (Doob's decomposition)

- $\forall$ stochastic process $(X_n)_{n=0}^{\infty}$ adapted to a filtration $(\mathcal{F}_n)_{n=0}^{\infty}$ can be decomposed as:

$$X_n = M_n + H_n \quad \text{"compensator"}$$

where (M_n) is a martingale and (H_n) is predictable with $H_0 = 0$.

- This decomosition is unique (up to a.s.)

- (X_n) is a submartingale $\Leftrightarrow$ (H_n) is increasing a.s. ← SKIP

Existence: guided by Example (d), define recursively

$$H_0 := 0; \qquad H_{n+1} - H_n := \mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n], \quad n = 0, 1, 2, \dots \qquad (*)$$

- By induction, H_{n+1} is $\mathcal{F}_n$-mble $\Rightarrow$ predictable.
- Define $M_n := X_n - H_n \Rightarrow \mathcal{F}_n$-mble;

$$\mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] = \mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] - \underbrace{\mathbb{E}[H_{n+1} - H_n \mid \mathcal{F}_n]}_{= H_{n+1} - H_n} \overset{(*)}{=} 0 \qquad (**)$$

($\mathcal{F}_n$-mble)

$\Rightarrow (M_n)$ is a martingale.

- Uniqueness follows from $(**)$: if $X_n = M_n + H_n \Rightarrow M_n = X_n - H_n$ $\Rightarrow$ $(**)$ must hold $\Rightarrow H_{n+1} - H_n$ must equal $\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]$.

- Last part follows from $(*)$.

The "compensator" method to construct mgles:

Let $S_n = Z_1 + \dots + Z_n$ be a simple random walk.

Ex (a) (Quadratic mgle revisited) Is S_n^2 a mgle? No:

$$\mathbb{E}[S_{n+1}^2 - S_n^2 \mid \mathcal{F}_n] = \mathbb{E}[2S_n Z_{n+1} + Z_{n+1}^2 \mid \mathcal{F}_n]$$

($S_{n+1}^2 = (S_n + Z_{n+1})^2$)

$$= 2S_n \underbrace{\mathbb{E}[Z_{n+1}]}_{0} + \underbrace{\mathbb{E}[Z_{n+1}^2]}_{1} = 1 = H_{n+1} - H_n$$

$\Rightarrow H_n = n.$

Doob $\Rightarrow$ $S_n^2 = M_n + n$ $\Rightarrow$ $M_n = S_n^2 - n$ is a mgle

(b) (Qubic mgle)

$$\mathbb{E}[S_{n+1}^3 - S_n^3 \mid \mathcal{F}_n] = \mathbb{E}[3S_n^2 Z_{n+1} + 3S_n Z_{n+1}^2 + Z_{n+1}^3 \mid \mathcal{F}_n]$$

($S_{n+1}^3 = (S_n + Z_{n+1})^3$)

$$= 3S_n^2 \underbrace{\mathbb{E}[Z_{n+1}]}_{0} + 3S_n \underbrace{\mathbb{E}[Z_{n+1}^2]}_{1} + \underbrace{\mathbb{E}[Z_{n+1}^3]}_{0}$$

$$= 3S_n = H_{n+1} - H_n$$

$$\Rightarrow H_n = \sum_{k=0}^{n-1} 3S_k$$

Doob $\Rightarrow$ $S_n^3 = M_n + \sum_{k=0}^{n-1} 3S_k$ $\Rightarrow$ $M_n = S_n^3 - \sum_{k=0}^{n-1} 3S_k$ is a mgle

Remark $S_{n+1}^3 - S_n^3 \approx$ derivative of S_n^3. $\longrightarrow$ Itô in 271B:

$$d(B_t^3) \overset{\text{Taylor}}{=} 3B_t^2 dB_t + \frac{1}{2} \cdot 6B_t \underbrace{(dB_t)^2}_{\overset{\text{Itô}}{=} dt} = 3B_t^2 dB_t + 3B_t dt$$

$$\Rightarrow B_t^3 = \underbrace{\int_0^t 3B_s^2 dB_s}_{M_t \text{ mgle}} + \int_0^t 3B_s ds$$

cover after remark on p. 169 ("def" of Itô)

MARTINGALE TRANSFORM

Prop Let $(X_n)_0^\infty$ be a martingale, $(H_n)_1^\infty$ be a predictable sequence. Then

$$(H\circ X)_0 := 0 \; ; \qquad (H\circ X)_n := \sum_{i=1}^{n} H_i \underbrace{(X_i - X_{i-1})}_{\text{martingale differences}} ,$$

is a martingale.

Proof $\mathbb{E}\left[(H\circ X)_{n+1} - (H\circ X)_n \,\middle|\, \mathcal{F}_n\right] = \mathbb{E}\left[\underbrace{H_{n+1}}_{\mathcal{F}_n\text{-mble}} (X_{n+1} - X_n) \,\middle|\, \mathcal{F}_n\right]$

$= H_{n+1} \cdot \mathbb{E}\left[X_{n+1} - X_n \,|\, \mathcal{F}_n\right] = 0$ ┘

Example: a gambling strategy

X_n = net amount of money we win at time n if we bet \$1 each time

H_n = our bet at time n (predictable ⇒ may depend on ∀ happening before)

$\Rightarrow (H\circ X)_n$ = our winnings at time n

Prop ⇒ In a zero-net game (X_n) no matter how good a betting strategy (H_n), the expected winnings are 0.

"You can't beat the system"

Remark Martingale transform is at the heart of the construction of Itô integral

$$\int_0^T \underset{\substack{\uparrow\\ \forall\text{ adapted process}}}{Z_t}\, d\underset{\substack{\uparrow\\ \text{Brownian motion}}}{B_t} = \lim_{\substack{\|\pi_n\|\to 0}} \underbrace{\sum_{i=1}^{n} Z_{t_{i-1}} (B_{t_i} - B_{t_{i-1}})}_{= (Z\circ B)_n}$$

$\pi_n = \{t_1, \dots, t_n\} \subset [0,T]$

Ex (Quadratic variation) Let (M_n) be $\forall$ martingale.

$$M_n^2 - M_{n-1}^2 = (M_{n-1} + \underbrace{\Delta M_n}_{M_n - M_{n-1}})^2 - M_{n-1}^2 = 2M_{n-1}\,\Delta M_n + (\Delta M_n)^2$$

telescoping $\Rightarrow$ $M_n^2 = \sum_{k=1}^{n} \underbrace{2M_{k-1}}_{\text{predictable}} \Delta M_k + \underbrace{\sum_{k=1}^{n} (\Delta M_k)^2}_{[M]_n}$

The first sum $\Downarrow$ Thm: mgle. $[M]_n$: quadratic variation

$\Rightarrow$ $M_n^2 - [M]_n$ is a mgle.

$$\left(d(M_t^2) = 2M_t\,dM_t + (dM_t)^2 \right)$$

(6) (Cubic mgle) Let $M_n = X_1 + \cdots + X_n$ be a simple r. walk.

$$M_n^3 - M_{n-1}^3 = (M_{n-1} + \Delta M_n)^3 - M_{n-1}^3 = 3M_{n-1}^2\,\Delta M_n + 3M_{n-1}\underbrace{(\Delta M_n)^2}_{1} + \underbrace{(\Delta M_n)^3}_{\Delta M_n}$$

($\Delta M_n = \pm 1$)

$$\Rightarrow M_n^3 = \underbrace{\sum_{k=1}^{n} 3M_{n-1}^2\,\Delta M_n}_{\text{mgle (transform Thm)}} + 3\sum_{k=1}^{n} M_{n-1} + \underbrace{M_n}_{\text{mgle}}$$

$\Rightarrow$ same result as in p.168.5 : $M_n^3 - 3\sum_{k=1}^{n} M_{n-1}$ is a mgle.

SKIP/HW?

Remark: The converse is given by Martingale Representation Thm

It can't hold in full generality even for $n=1$:

($M_1 = a + b(X_1 - c)$ $\Rightarrow$ M_1 must be linear in X_1, which is not true in general) ← could be a HW problem;

– but is true e.g. for Rademacher (see Khoshnevisan or Williams 15.1)

& Gaussian process (see MRT for Itô)

STOPPING TIMES

Def A r.variable $T \in \mathbb{N} \cup \{\infty\}$ is a stopping time w.r.to a filtration $(\mathcal{F}_n)$ if

$$\{T \le n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$$

$\Updownarrow$ (complement)

$$\{T > n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$$

$\Updownarrow$

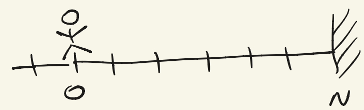
$$\{T = n\} \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$$

$\left[\{T=n\} = \underset{\in \mathcal{F}_n}{\{T \le n\}} \setminus \underset{\in \mathcal{F}_{n-1} \subset \mathcal{F}_n}{\{T \le n-1\}} \in \mathcal{F}_n \right]$

Examples: (a) constant, e.g. $T = 10$.

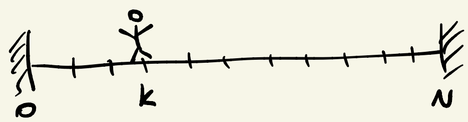
(b) let (S_n) be a simple r.walk. Hitting time of a given point N:

$$T := \min\{n : S_n = N\}$$



(c) Gambler's ruin: let (S_n) be a simple r.walk starting at k.
Exit time of a given interval $[0, N]$:

$$T := \min\{n : S_n \in \{0, N\}\}.$$



(d) Play until either won \$100 or have played 10 games:

$$T = \min\{n : X_n = 100\} \wedge 10$$

(e) Playing until reach absolute max is NOT a stopping time.

Prop (Stopped martingale) If (X_n) is a martingale and T is a stopping time then $(X_{n\wedge T})$ is a martingale.

$$X_{n\wedge T} - X_0 = \sum_{i=1}^{n\wedge T}(X_i - X_{i-1}) = \sum_{i=1}^{n} \mathbb{1}_{\{i\le T\}}(X_i - X_{i-1})$$

$\mathbb{1}_{\{i\le T\}}$ is $\mathcal{F}_{i-1}$-mble, since $\{T\ge i\} = \{T > i-1\} \in \mathcal{F}_{i-1}$
$\Rightarrow$ predictable.

Prop. 169 $\Rightarrow$ $(X_{T\wedge n} - X_0)$ is a martingale.
Since (X_0) is trivially a martingale, $(X_{T\wedge n})$ is a martingale. $\square$

Mention local martingales?

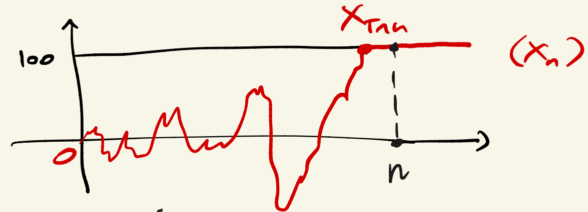
Remark Prop $\Rightarrow$ $\mathbb{E}X_{n\wedge T} = \mathbb{E}X_0 \quad \forall n\in\mathbb{N}$

$\Rightarrow$ NO WINNING STOPPING STRATEGY IN PRESENCE OF A FIXED DEADLINE n

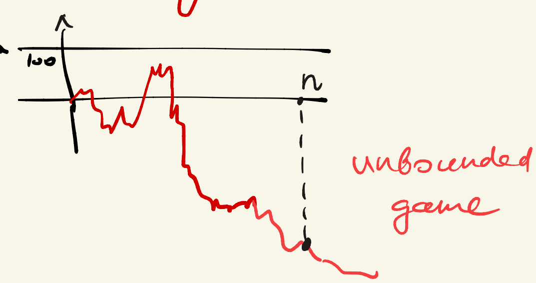
Example: stop when reached \$100:

$$T = \min\{n : X_n = 100\}$$

Prop $\Rightarrow$ $\mathbb{E}X_{n\wedge T} = X_0 = 0$ HOW?!



- If $T\le n$, goal is reached by deadline n; $\Rightarrow X_{T\wedge n} = 100$
- But if $T>n$, goal is NOT reached by deadline →
  $X_{T\wedge n}$ can be very negative (debt)



These two counterweigh each other.

Ex (Optional switching): HW (270B 2019 p. 93)