Without fixed deadline?

## OPTIONAL STOPPING

- let $(X_n)$ be a martingale and $T$ be a stopping time.

Q: is $\mathbb{E}X_T > \mathbb{E}X_0$ possible? (stopping strategy for making money)

Ans: sometimes. Examples:

(a) Symmetric random walk with sticky wall at 1,

i.e. $T = \min\{n : X_n = 1\}$.

$X_T = 1$ deterministically $\Rightarrow \mathbb{E}X_T > \mathbb{E}X_0$

(b) St. Petersburg martingale $\left( p.163: \ X_0 = 1, \ X_n | X_{n-1} = \begin{cases} 2X_{n-1}, & \text{prob } 1/2 \\ 0, & \text{prob } 1/2 \end{cases} \right)$

Wait until broke: $T = \min\{n : X_n = 0\} \sim \text{Geom}(1/2)$

$\Rightarrow X_T = 0$ deterministically $\Rightarrow \mathbb{E}X_T < \mathbb{E}X_0$

However:

**THM** (Doob's optional stopping thm)

Let $(X_n)$ be a martingale and $T$ be a stopping time.

Assume $\exists\, C>0$ s.t. at least one of the following conditions hold:

(i) $T<\infty$ a.s. and $|X_{n\wedge T}|<C$ a.s. $\forall n$ ← bounded mgle

(ii) $T<C$ a.s. ← bounded time

(iii) $\mathbb{E}T<\infty$ and $\forall n\in\mathbb{N}$: $\begin{cases} \mathbb{E}\left[|X_{n+1}-X_n| \mid \mathcal{F}_n\right]<C \text{ a.s.} \\ \text{on the event } \{T>n\},\ \forall n\in\mathbb{N} \end{cases}$ ← limited bets

Then $\mathbb{E}X_T=\mathbb{E}X_0$.

**Proof** (i) $T<\infty$ a.s. $\Rightarrow$ $X_{n\wedge T}\xrightarrow{\text{a.s.}} X_T$ $(n\to\infty)$

$$\left.\begin{array}{l} \text{D.C.T} \Rightarrow \underbrace{\mathbb{E}X_{n\wedge T}}_{\substack{\text{stopped mtgale } \| \text{ (p.171)} \\ \mathbb{E}X_0}} \to \mathbb{E}X_T \end{array}\right\} \Rightarrow \square.$$

(ii) $$|X_{n\wedge T}|=\left|X_0+\sum_{i=1}^{n\wedge T}(X_i-X_{i-1})\right| \overset{\text{a.s.}}{\leq} |X_0|+\sum_{i=1}^{T}|X_i-X_{i-1}| =: M$$

$T\leq C$ a.s. $\Rightarrow$ $(X_{n\wedge T})$ is dominated by an integrable r.v. $M$.

Apply DCT as in (i).

(iii) $$M=|X_0|+\sum_{i=1}^{\infty}\mathbb{1}_{\{i\leq T\}}|X_i-X_{i-1}|.$$

$$\Rightarrow \mathbb{E}M=\mathbb{E}|X_0|+\sum_{i=1}^{\infty}\mathbb{E}\Big[\mathbb{E}\big[\underbrace{\mathbb{1}_{\{T\geq i\}}}_{\substack{\| \\ \mathbb{1}_{\{T>i-1\}} \text{ is } \mathcal{F}_{i-1}\text{-mble}}}|X_i-X_{i-1}| \mid \mathcal{F}_{i-1}\big]\Big]$$

$$=\mathbb{E}|X_0|+\sum_{i=1}^{\infty}\mathbb{E}\Big[\underbrace{\mathbb{1}_{\{T\geq i\}}\,\mathbb{E}\big[|X_i-X_{i-1}| \mid \mathcal{F}_{i-1}\big]}_{\substack{\wedge\!\| \\ C\cdot\mathbb{1}_{\{T>i-1\}} \text{ (by assm.)}}}\Big]$$

$$\leq \mathbb{E}|X_0|+C\underbrace{\sum_{i=1}^{\infty}P\{T\geq i\}}_{\substack{\| \\ \mathbb{E}T<\infty \text{ (integrated tail formula)}}}$$

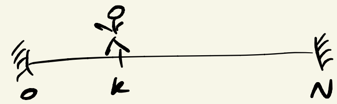
$<\infty$.

Apply DCT as before. $\square$

## EXAMPLES on simple random walk $(S_n)$.

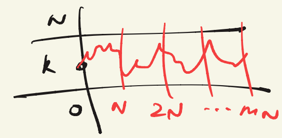
(a) Hitting time of $\{0,N\}$, starting at $S_0 = k$ (gambler's ruin)

$$T := \min\{n : S_n \in \{0,N\}\}$$




**Prop** $\mathbb{E}T < \infty$

Break time into blocks of length $N$. Each block has the prob. $2^{-N}$ of absorbtion, independently. Formally: if $T > mN$ then the walk's range must be $\leq N$ over each of the $m$ blocks:




$$|S_N - S_0| < N, \quad |S_{2N} - S_N| < N, \ \ldots, \quad |S_{mN} - S_{(m-1)N}| < N$$

Indep. events, each with prob. $\geq 1 - 2^{-N}$ $\Rightarrow$

$$\mathbb{P}\{T > mN\} \leq (1 - 2^{-N})^m, \quad m \in \mathbb{N}$$

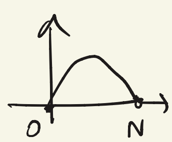
Integrated tail $\Rightarrow$ $\mathbb{E}[T/N] \leq \sum_{m=1}^{\infty} \mathbb{P}\{T > mn\} < \infty$ (geometric series)

- **Prop** $\mathbb{P}\{S_T = N\} = \frac{k}{N}$, $\mathbb{P}\{S_T = 0\} = 1 - \frac{k}{N}$, recovering the result on p. 146.

- O.S.T. (i) $\Rightarrow$ $\mathbb{E}S_T = \mathbb{E}S_0 = k$.

$S_T \in \{0,N\}$ $\quad \mathbb{E}S_T = N \cdot \mathbb{P}\{S_T = N\}$.

- **Prop** $\mathbb{E}T = k(N-k)$




ALTERNATIVE PROOF

$\mathbb{E}\left[S_{n\wedge T}^2 - (n \wedge T)\right] = \mathbb{E}[S_0^2 - 0] = k^2$ (mgle)

DCT $\Rightarrow$ $\mathbb{E}[S_{n\wedge T}^2] \xrightarrow{\text{since } T<\infty \text{ a.s.}} \mathbb{E}S_T^2 = Nk$

$\Rightarrow \mathbb{E}[n \wedge T] \to Nk - k^2 = k(N-k)$

MCT $\downarrow$

$\mathbb{E}[T]$. $\square$

Consider the quadratic martingale $X_n := S_n^2 - n$, apply O.S.T. (iii)

(assm. (iii) holds since, On the event $\{T > n\}$, $S_n \in (0,N)$ and $S_{n+1} \in [0,N]$
$\Rightarrow |X_{n+1}^2 - X_n^2| = |S_{n+1}^2 - S_n^2 + 1| \leq 2N+1$)

$$\Rightarrow \mathbb{E}[S_T^2 - T] = \mathbb{E}[S_0^2 - 0] = k^2 \quad \Rightarrow \quad \mathbb{E}[T] = \underbrace{\mathbb{E}[S_T^2]}_{\substack{\| \text{ Prop. above} \\ N^2 \cdot \frac{k}{N}}} - k^2 = k(N-k)$$

(b) Hitting time of 1, starting at $S_0 = 0$: $T := \min\{n : S_n = 1\}$.

**Prop** $\mathbb{E}T = \infty$ (on ave, wait $\infty$ time to get \$1!)

$\ulcorner$ N If $\mathbb{E}T < \infty$, then OST (iii) would yield $\mathbb{E}S_T = \mathbb{E}S_0 = 0$.
But $\mathbb{E}S_T = 1$ by def of $T$. $\lightning$ $\lrcorner$

HOWEVER:

**Prop** $T < \infty$ a.s.

$\ulcorner$ Fix $\forall \lambda > 0$ and consider the product mgle:

$$X_n := \prod_{k=1}^{n} \frac{e^{\lambda X_k}}{\underbrace{\mathbb{E}e^{\lambda X_k}}_{\substack{\| \, X\sim Rad \\ \frac{e^{\lambda}+e^{-\lambda}}{2} = \cosh(\lambda)}}} = \frac{\exp(\lambda S_n)}{\cosh(\lambda)^n}.$$

- The stopped mgle:

$$X_{n\wedge T} = \frac{\exp(\lambda S_{n\wedge T})}{\cosh(\lambda)^{n\wedge T}} \xrightarrow{n\to\infty} \frac{\exp(\lambda S_T)}{\cosh(\lambda)^T} \cdot \mathbb{1}_{\{T<\infty\}} + 0 \cdot \mathbb{1}_{\{T=\infty\}}$$

($S_n \le 1$ & $\cosh(\lambda) > 1$)

& uniformly bounded: $0 \le X_{n\wedge T} \le \exp(\lambda \cdot 1)$ (both when $T<\infty$ & $T=\infty$)

- $\overset{DCT}{\Longrightarrow}$ $\mathbb{E}X_{n\wedge T} \xrightarrow{n\to\infty} \mathbb{E}\left[\frac{\exp(\lambda \overset{\overset{1}{\|}}{S_T})}{\cosh(\lambda)^T} \cdot \mathbb{1}_{\{T<\infty\}}\right]$

mgle $\|$

$\mathbb{E}X_0 = 1$. Thus, dividing both sides by $e^{\lambda}$ $\Rightarrow$

$$\mathbb{E}\big[\underbrace{\cosh(\lambda)^{-T} \mathbb{1}_{\{T<\infty\}}}_{\le 1}\big] = e^{-\lambda} \qquad \forall \lambda > 0$$

☺

- Let $\lambda \downarrow 0$. Then $\cosh(\lambda)^{-T} \to 1$ and $e^{-\lambda} \to 1$. DCT $\Rightarrow$

$$\mathbb{E}\,\mathbb{1}_{\{T<\infty\}} = 1$$
$$\|$$
$$\mathbb{P}\{T<\infty\}$$

$\lrcorner$

- Distribution of $T$?

Keep exploring the product mgle $(X_n)$. ☺ & $T<\infty$ a.s. $\Rightarrow$

$$\mathbb{E}\left[\cosh(\lambda)^{-T}\right]=e^{-\lambda} \quad \forall \lambda>0 \qquad (**)$$

Change var's to $\frac{1}{a}=\cosh(\lambda)=\frac{e^{\lambda}+e^{-\lambda}}{2} \in [1,\infty) \Rightarrow a\in(0,1]$

Solving $\Rightarrow e^{-\lambda}=\frac{1\pm\sqrt{1-a^2}}{a}$ $\quad e^{-\lambda}\leq 1$ for $\lambda>0 \Rightarrow$ "$-$" sign is valid.

$\overset{(**)}{\Longrightarrow}$ $\mathbb{E}[a^T]=\frac{1-\sqrt{1-a^2}}{a}$ "generating function" of $T$ $\qquad (\circ)$

$\mathbb{E}[a^T] = \sum_{k=1}^{\infty} a^k P\{T=k\}$

$\frac{1-\sqrt{1-a^2}}{a} = \sum_{n=1}^{\infty}\frac{1}{2n-1}\binom{2n}{n}2^{-2n}\cdot a^{2n-1}$ Taylor series $(a\downarrow 0)$

Matching the coefficients $\Rightarrow$

**HW**

- Compute the distr. of hitting time of $\{-N,N\}$ starting at 0

**THM** (Hitting time of 1)

$$P\{T=2n-1\}=\frac{1}{2n-1}\binom{2n}{n}2^{-2n}, \quad n\in\mathbb{N}$$

$$\approx\frac{1}{2\sqrt{\pi}}n^{-3/2}, \quad n\to\infty \quad \Rightarrow \text{ even } \mathbb{E}\sqrt{T}=\infty$$

**Cor** (Recurrence of a 1D random walk) The time to get back to 0, $T=\min\{n>0: S_n=0\}$, is finite a.s.

⌈Condition on 1st step. If $-1$, $T=$ time(to get from $-1$ to $0$) $<\infty$ a.s. (Prop)
If $1$, $T=$ time(to get from $1$ to $0$) $<\infty$ a.s. (Prop).⌋

Heuristic proof of $(\circ)$: condition on 1st step $\Rightarrow$

$$x:=\mathbb{E}[a^T]=\mathbb{E}\underbrace{[a^T|S_1=-1]}_{\overset{\mathcal{D}}{=}a^{1+T_1+T_2}}\cdot\tfrac{1}{2}+\mathbb{E}\underbrace{[a^T|S_1=1]}_{=a^1}\cdot\tfrac{1}{2}=a\cdot\left(\mathbb{E}[a^T]\right)^2+\frac{a}{2}$$

$T_1 \perp\!\!\!\perp T_2 \sim T$ (strong Markov prop.)

$\Rightarrow x=\frac{ax^2}{2}+\frac{a}{2}$. Solve $\Rightarrow(\circ)$.

ⓒ **Wald's equation**

Let $Z_1, Z_2, \dots$ be iid r.v's with finite mean $\overset{\mu}{\shortparallel}$, $T$ be a stopping time with $ET < \infty$. Then

$$\mathbb{E}\left[\sum_{i=1}^{T} Z_i\right] = \mathbb{E}[T]\cdot\mathbb{E}[Z_1]$$

Proof $S_0 := 0$, $S_n := Z_1 + \dots + Z_n$ $\Rightarrow$ $X_n := S_n - n\mu$ is a martingale

Then OST (iii) applies

$$\Big( \mathbb{E}\big[\underbrace{|X_{n+1} - X_n|}_{= Z_{n+1}-\mu \text{ is } \mathcal{F}_n\text{-mble}} \,\big|\, \mathcal{F}_n\big] = \mathbb{E}|Z_{n+1} - \mu| \le \mathbb{E}|Z_1| + |\mu| =: C < \infty \Big)$$

$$\Rightarrow \mathbb{E}\underbrace{X_T}_{= S_T - T\mu} = \mathbb{E}X_0 = 0 \quad \Rightarrow \quad \mathbb{E}S_T = \mathbb{E}[T]\mu. \qquad \square$$

**HW**: Show by example that independence assumption can't be removed:

$Z_n = Z \sim$ Rademacher; $T = \begin{cases} 2 & \text{if } Z = 1 \\ 1 & \text{if } Z = -1 \end{cases}$

An alternative argument for Prop in (6):

If $ET < \infty$ then, by Wald's equation,

$$\mathbb{E}\Big[\sum_{i=1}^{T} \overset{\text{iid Rademacher}}{Z_i}\Big] = \mathbb{E}[T]\cdot\underbrace{\mathbb{E}[Z_1]}_{=0} = 0. \quad \text{But } \sum_{i=1}^{T} Z_i = 1 \text{ deterministically.} \ \lightning$$