A.s. martingale convergence

- Do martingales converge: $X_n \xrightarrow{a.s.} X_\infty$ to some r.v. X_∞?
- Some do (r. walk with 1 or 2 absorbing walls, St. Pete's mgle)
- Some don't (simple r. walk). (check formally!)
- L^1-bounded mgles always do:

THM [A.s. Martingale Convergence Thm]

Let (X_n) be a martingale s.t. $\sup_n \mathbb{E}|X_n| < \infty$.

Then (X_n) converges a.s. to some r.v. X_∞, and $\mathbb{E}|X_\infty| < \infty$.

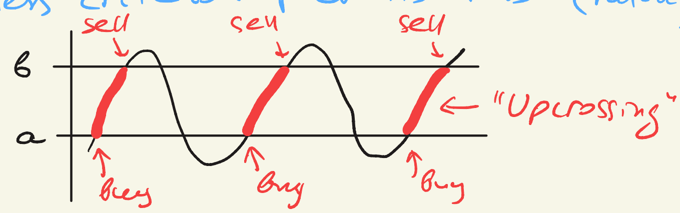
Heuristic proof:

There are 2 reasons a sequence (x_n) fails to converge:

(a) it goes off to $+\infty$ or $-\infty$.

But the L^1 boundedness criterion prevents this (Fatou)

(b) it oscillates:



$\Rightarrow$ we can get $\forall$ rich by "buy low, sell high" strategy

But this contradicts the "can't beat the system" thm (p. 169)

Let's formalize (b):

Def (Upcrossings) Fix a sequence $x_0, x_1, \dots$ and numbers $a < b$.

An upcrossing is an interval $[s,t]$ where $x_s < a < b < x_t$.

$U_n(a,b) := \#(\text{disjoint upcrossings by time } n)$.

- A mgle has finite upcrossings in expectation:

Lem (Upcrossing Inequality) Let (X_n) be a martingale. Then $\forall\, a<b$, $\forall\, n \in \mathbb{N}$:

$$(b-a)\cdot \mathbb{E}\left[U_n(a,b)\right] \leq \mathbb{E}(X_n-a)^- \qquad \left(x = \underset{\geq 0}{x^+} - \underset{\geq 0}{x^-}\right)$$

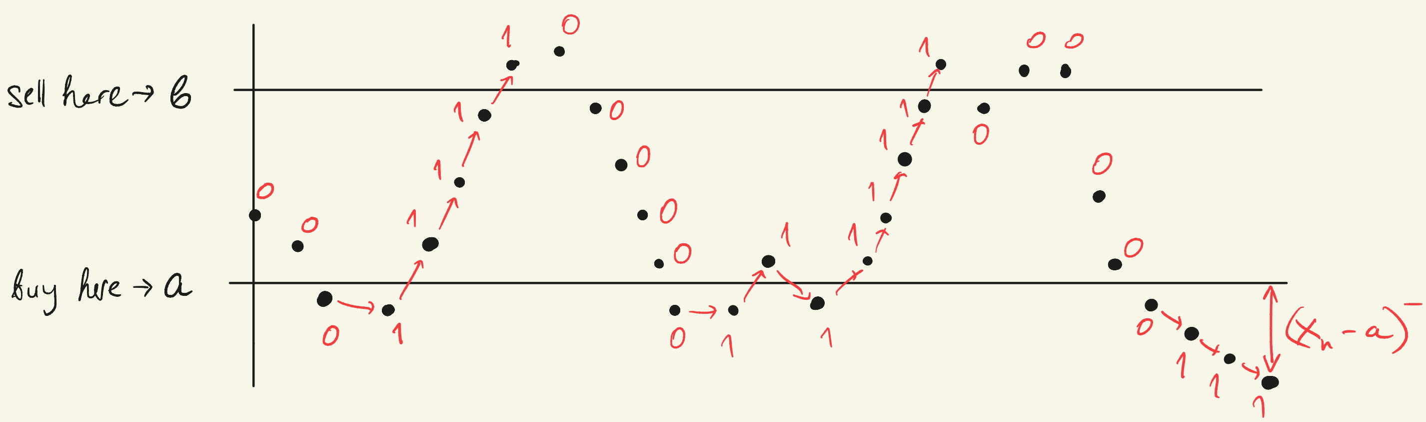
Define the strategy using the "hold indicators":

$$H_1 := \mathbb{1}\{X_0<a\}$$

keep holding   Buy

$$H_n := \mathbb{1}\{H_{n-1}=1 \text{ and } X_{n-1}\leq b\} + \mathbb{1}\{H_{n-1}=0 \text{ and } X_{n-1}<a\}$$

(H_n) is predictable (by induction).



$\Rightarrow$ martingale transform satisfies

$$(H\circ X)_n = \sum_{i=1}^{n} H_i(X_i - X_{i-1}) \geq (b-a)\,U_n(a,b) - (X_n-a)^- \qquad (*)$$

($\forall$ upcrossing contributes $(b-a)$;
the last (incomplete) upcrossing creates a loss $\leq (X_n-a)^-$)

- $(H\circ X)_n$ is a martingale (Prop p.169) & $(H\circ X)_0 = 0$ (def) $\Rightarrow$

$$\mathbb{E}(H\circ X)_n = 0.$$

Take $\mathbb{E}$ in $(*)$ $\Rightarrow$ $\square$.

⇒ **Lem** (**L^1-bounded Martingales don't oscillate**)

Let (X_n) be a martingale s.t. $\sup_n \mathbb{E}|X_n| < \infty$.

Then $$\mathbb{P}\left\{ U_n(a,b) < \infty \quad \forall\ a<b \right\} = 1.$$

Q: Is there a quantitative bound: $\mathbb{E} \sup_{a<b} (a-b)\, U_n(a,b) \le ?$

① Fix $a<b$. Upcrossing lemma ⇒

$$\mathbb{E} U_n(a,b) \le \frac{\mathbb{E}|X_n - a|}{|b-a|} \le \frac{\mathbb{E}|X_n| - a}{|b-a|}$$

⇒ by assumption, this yields $\sup_n \mathbb{E} U_n(a,b) < \infty$.

$U_n(a,b)$ increase in n ⇒ $\exists\ U_\infty(a,b) := \lim_n U_n(a,b) \in \mathbb{N} \cup \{\infty\}$.

Fatou ⇒ $\mathbb{E} U_\infty(a,b) \le \liminf_n \mathbb{E} U_n(a,b) < \infty$

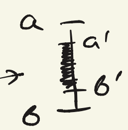
⇒ $U_\infty(a,b) < \infty$ a.s.

②. By a union bound, $\mathbb{P}\{ U_\infty(a,b) < \infty\ \ \forall$ rational $a<b\} = 1$.

③. If $U_\infty(a,b) = \infty$ for some real $a<b$

then $U_\infty(a',b') = \infty$ for some rational $a'<b'$,

but this happens with prob. 0.



Proof of a.s. MaConv Thm

① $\lim_n X_n$ exists a.s. (could be $\pm\infty$):

N: $\liminf_n X_n < \limsup_n X_n$ ⇒ $\exists\ a<b$: $\liminf_n X_n < a < b < \limsup_n X_n$

⇒ $X_n < a$ for ∞ many n & $X_n > b$ for ∞ many n $\Big\}$ ⇒ $U_n(a,b) = \infty$, but this happens with prob. 0 (Lem)

② Thus $X_n \to X_\infty$ a.s. for some r.v. $X_\infty \in \mathbb{R} \cup \{\pm\infty\}$.

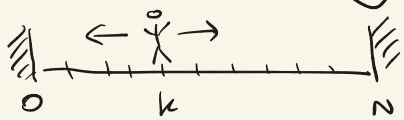
Fatou ⇒ $\mathbb{E}|X_\infty| \le \liminf_n \mathbb{E}|X_n| < \infty$ by assumption. ☐.

Examples

HW

(a) Simple random walk — thm does not apply ($\mathbb{E}|S_n| \to \infty$)

(b) R. walk with absorbing walls @ 0 and N, starting at k



is a bounded mgle. a.s. Ma Conv Thm $\Rightarrow$

$$X_n \xrightarrow{a.s.} X_\infty \overset{a.s.}{\in} \{0, N\}$$

we showed that $T < \infty$ a.s. $\Rightarrow$ with prob. 1, the walk eventually hits either 0 or N, and stays there

• mgle $\Rightarrow$ $\mathbb{E}X_n = \mathbb{E}X_0 = k \quad \forall n$

DCT $\Rightarrow$ $\mathbb{E}X_\infty = k$. $\Rightarrow$ $X_\infty = \begin{cases} 0, & \text{prob } k/N \\ N, & \text{prob } 1 - k/N \end{cases}$

giving an alternative sol. to this problem.

(c) St. Petersburg mgle: $X_n = \begin{cases} 2^n, & \text{prob } 2^{-n} \\ 0, & \end{cases}$

$\mathbb{E}|X_n| = 1 \Rightarrow$ a.s. Ma Conv Thm applies $\Rightarrow$ $X_n \xrightarrow{a.s.} X_\infty$.

$X_\infty = 0$ (by Borel-Cantelli)

SKIP

(d) More generally, for $\forall$ product mgle

[Kakutani — see Williams]

$$Y_n = Z_1 \cdots Z_n, \quad \mathbb{E}Z_i = 1, \quad Z_i \geq 0$$

MaCT $\Rightarrow$ $Y_n \xrightarrow{a.s} Y_\infty$

Either $Y_\infty = 1$ a.s (if $Z_1 = 1$ a.s.) or $Y_\infty = 0$ a.s.

(prove it! Hint: take $\log(\cdot)$ & use SLLN.

(e) Branching process (Galton-Watson) size of population $\mu =$ mean offspring #

$\mathbb{E}[Z_n] = \mu^n$.

$\Rightarrow$ a.s. Ma Conv thm applies to the mgle Z_n/μ_n

$$\Rightarrow \quad \frac{Z_n}{\mu_n} \xrightarrow{a.s.} W \quad \text{some r.v.}$$

HW $\mu < 1 \Rightarrow W \equiv 0$ (check!) (extinction)

(f) Polya's urn : $X_n =$ fraction of white Balls < 1

$$\Rightarrow \quad X_n \xrightarrow{a.s.} X_\infty \quad \text{some r.v.}$$

Actually, $X \sim Beta(N, M) \sim Unif\,[0,1]$ if $N = M = 1$.

#white #Black Balls initially

APPLICATION: FISHER-WRIGHT MODEL IN POPULATION GENETICS

- Start with a population of n_0 individuals, with $p_0 n$ of them being type A and $(1-p_0)n$ type B.
- Randomly select n_1 individuals with replacement.
  Each selected individual produces one offspring of the same type as its parent.
  Those n_1 offspring form the next generation, and the process repeats giving generations of size $n_1, n_2, \dots$

THM (We will all become the same) Assume the population size grows sublinearly: $n_t / t \to 0$ as $t \to \infty$. Then, with prob. 1, the population eventually becomes homogeneous, i.e. after some time all individuals will be of the same type (A with prob. p_0, B with prob. $1-p_0$)

Γ. $X_t := \#\{\text{individuals of type A in generation } t\}$ ⇒

←more rigorously, $X_{t+1} \mid p_1, \dots, p_t$

$$\left\{ \begin{array}{l} X_{t+1} \mid p_t \sim \text{Binom}(n_{t+1}, p_t) \\ p_{t+1} := \dfrac{X_{t+1}}{n_{t+1}} \end{array} \right\} \qquad t = 0, 1, 2, \dots$$

- (p_t) is a martingale:

$$\mathbb{E}[p_{t+1} \mid p_t] = \frac{1}{n_{t+1}} \mathbb{E}[\overbrace{X_{t+1}}^{\sim \text{Binom}(n_{t+1}, p_t)} \mid p_t] = \frac{1}{n_{t+1}} \cdot n_{t+1} p_t = p_t \qquad (*)$$

- a.s. Ma Conv Thm ⇒ $p_t \xrightarrow{a.s} p_\infty \qquad (\circ)$

• $h_t := p_t(1-p_t)$ measures fixation/homogenization

If $X \sim \mathrm{Binom}(n,p)$ then
$\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}X)^2$
$= np(1-p) + n^2p^2$

<u>CLAIM</u>: $\mathbb{E}h_{t+1} = \left(1 - \frac{1}{n_{t+1}}\right)\mathbb{E}h_t$

$$\Gamma\ \underbrace{\mathbb{E}[h_{t+1} \mid p_t]}_{\parallel\ p_{t+1} - p_{t+1}^2} = \underbrace{\mathbb{E}[p_{t+1} \mid p_t]}_{\overset{(*)}{\parallel}\ p_t} - \mathbb{E}[p_{t+1}^2 \mid p_t] = p_t - \frac{1}{n_{t+1}^2}\underbrace{\mathbb{E}[X_{t+1}^2 \mid p_t]}_{\parallel\ n_{t+1}p_t(1-p_t) + n_{t+1}^2 p_t^2}$$

$$= p_t - \frac{p_t(1-p_t)}{n_{t+1}} - p_t^2 = \left(1 - \frac{1}{n_{t+1}}\right)\underbrace{p_t(1-p_t)}_{\parallel\ h_t}. \quad \text{Take } \mathbb{E} \Rightarrow \square$$

$$\bullet \Rightarrow \mathbb{E}h_t = \left[\prod_{k=1}^{t}\underbrace{\left(1 - \frac{1}{n_{k+1}}\right)}_{\wedge\!\wedge\ e^{-1/n_k}}\right]\underbrace{\mathbb{E}h_0}_{\wedge\!\wedge\ 1} \le \exp\left(-\underbrace{\sum_{k=1}^{t}\frac{1}{n_k}}_{\Downarrow\ \log t}\right) < \frac{1}{t^{10}} \quad \text{for large } t$$

$$\frac{1}{n_k} \overset{\text{assm}}{\ge} 1/k$$

$$\bullet \Rightarrow \mathbb{P}\left\{h_t \ge \frac{1}{n_t^2}\right\} \overset{\text{Markov}}{\le} \frac{n_t^2}{t^{10}} \overset{\text{assm}}{\le} \frac{1}{t^8} \quad \text{is summable}$$

$$\xrightarrow{\text{BCI}} \mathbb{P}\left\{h_t \ge \frac{1}{n_t^2} \text{ i.o.}\right\} = 0$$

i.e. with prob. 1, eventually (i.e. $\exists t_0\ \forall t > t_0$) we have

$$n_t p_t \cdot n_t(1-p_t) = n_t^2 h_t < 1 \Rightarrow \underbrace{n_t p_t}_{\parallel\ X_t} < 1 \text{ or } \underbrace{n_t(1-p_t)}_{\parallel\ n_t - X_t} < 1.$$

$X_t \leftarrow$ integers $\rightarrow n_t - X_t$

$$\Rightarrow X_t \in \{0, n_t\} \Rightarrow p_t \in \{0,1\}.$$

• Combining with $(\circ)$ $\Rightarrow$ $p_\infty \in \{0,1\}$

$$\mathbb{P}\{p_t = 1 \text{ eventually}\} = \mathbb{P}\{p_\infty = 1\} = \mathbb{E}p_\infty \overset{\text{DCT}}{=\!=} \lim_t \mathbb{E}p_t \overset{\text{martingale}}{=\!=} p_0 \qquad \checkmark$$