L^1 Martingale Convergence

- Martingale Convergence Thm: if

$$\sup_n \mathbb{E}|X_n| < \infty \qquad (*)$$

then $X_n \xrightarrow{a.s.} X_\infty$.

Q1 Can we upgrade the a.s. convergence to the L^1-convergence?

Q2 Can we reconstruct the mgle from its limit:

$$X_n = \mathbb{E}[X_\infty | \mathcal{F}_n) \quad \forall n \quad ?$$

- Q1: Sometimes we can (when DCT applies)
- Sometimes we can't:

  (a). St. Petersburg mgle: $X_\infty = 0$

  (b) same for Galton-Watson process with $\mu < \infty$.

- We always can if $(*)$ is strengthened to:

Def (Uniform integrability)

A sequence of r.v's (X_n) is uniformly integrable if

$$\sup_n \mathbb{E}\left[|X_n| \mathbb{1}_{\{|X_n| > M\}}\right] \to 0 \quad \text{as } M \to \infty.$$

Examples: (a) $\forall X \in L^1$, $(X, X, X, \dots)$ is U.I.

$\lceil$ DCT $\Rightarrow \mathbb{E}\left[|X| \mathbb{1}_{\{|X| > M\}}\right] \to 0$ as $M \to \infty \rfloor$

(b) $\forall$ dominated, i.e. if $|X_n| < X$ with $\mathbb{E}X < \infty$ (by (a))

(c) NOT St. Petersburg mgle $X_n = \begin{cases} 2^n; & \text{prob} = 2^{-n} \\ 0; & \end{cases}$

• The events $\{|X_n|>M\}$ are "the worst" and can be replaced with $\forall$ other events F_n with $P(F_n) \to 0$:

Lem (Criterion of uniform integrability)

(X_n) is uniformly integrable if and only if:

(a) $\sup_n \mathbb{E}|X_n| < \infty$ and

(b) $\forall \varepsilon>0 \; \exists \delta>0 : P(F)<\delta \Rightarrow \sup_n \mathbb{E}\left[|X_n| \mathbb{1}_F\right] < \varepsilon.$

• Assm $\Rightarrow$ $\exists M: E_1 \le 1$ $\forall n$ $\Rightarrow \mathbb{E}X_n \le M+1$ $\forall n$

• $E_2 \le M$

WLOG $X_n \ge 0$ (otherwise $X_n \mapsto |X_n|$)

($\Rightarrow$) $$\mathbb{E}\left[X_n \mathbb{1}_F\right] = \underbrace{\mathbb{E}\left[X_n \mathbb{1}_{F\cap\{X_n>M\}}\right]}_{E_1} + \underbrace{\mathbb{E}\left[X_n \mathbb{1}_{F\cap\{X_n\le M\}}\right]}_{E_2}$$

• To prove (a), take $F=\Omega$. Assm $\Rightarrow$ $\exists M: E_1 \le 1;\; E_2 \le M$ $\Rightarrow$

$$\mathbb{E}X_n \le 1+M \quad \forall n$$

• To prove (b), $\forall \varepsilon>0 \;\exists M:$ $E_1 \le \frac{\varepsilon}{2}$ $\forall n$;

$E_2 \le M\,P(F) \le \varepsilon/2$ if $P(F) < \frac{\varepsilon}{2M} =: \delta$ $\Bigg\}$ $\Rightarrow$

$$\mathbb{E}\left[X_n \mathbb{1}_F\right] < \varepsilon \quad \forall n.$$

($\Leftarrow$) Fix $\forall \varepsilon>0$, choose $\delta>0$ as in (b).

$F := \{X_n > M\}$ satisfies

$$P(F) \overset{\text{Markov}}{\le} \frac{\mathbb{E}X_n}{M} \overset{(a)}{\le} \frac{C}{M} < \delta \qquad \forall M > \frac{C}{\delta}$$

$$\overset{(b)}{\Rightarrow} \mathbb{E}\left[X_n \mathbb{1}_F\right] < \varepsilon.$$

HW Does (a) follow from (b)? Prove or give a counterexample.

More examples of U.I. sequences

Lem 1 If (X_n) and (Y_n) are both U.I, then (X_n+Y_n) is U.I.

⌈ UI criterion & Δ ineq. ⌋

Lem 2 If $\sup_n \mathbb{E}|X_n|^p < \infty$ for some $p>1$ $\Rightarrow$ (X_n) is unif. intgble.

⌈ WLOG $X_n \geq 0$

$$\mathbb{E}\left[X_n \mathbb{1}\{X_n>M\}\right] = \mathbb{E}\left[X_n^p \underbrace{X_n^{1-p} \mathbb{1}\{X_n>M\}}_{\leq M^{1-p}}\right] \leq M^{1-p}\, \mathbb{E} X_n^p .$$

Take $\sup_n$ and let $M \to \infty$. ⌋

Lem 3 Let $X \in L^1$ and $(\mathcal{F}_n)$ be $\forall$ σ-algebras. Then the sequence $X_n := \mathbb{E}[X|\mathcal{F}_n]$ is uniformly integrable.

⌈ WLOG $X \geq 0$ (otherwise decompose $X = X^+ - X^-$ and use Lem above)

$$\underset{\text{WTS } \downarrow\ 0\ ?}{\mathbb{E}\left[X_n \mathbb{1}_{\{X_n>M\}}\right]} = \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_n]\, \underbrace{\mathbb{1}_{\{X_n>M\}}}_{\in\, \mathcal{F}_n}\right] \overset{\text{Tower}}{=\!=} \mathbb{E}\left[X \mathbb{1}_{\{X_n>M\}}\right]$$

- $(X, X, \dots)$ is U.I. (Ex. a) $\xRightarrow{\text{Criterion}}$ (or just a standard consequence of integrability):

$$\forall \varepsilon>0\ \exists \delta>0: \ P(F)<\delta \Rightarrow \mathbb{E}[X \mathbb{1}_F] < \varepsilon$$

- $P\{X_n>M\} \overset{\text{Markov}}{\leq} \dfrac{\mathbb{E}X_n}{M} \overset{\text{Tower}}{=\!=} \dfrac{\mathbb{E}X}{M} < \delta \quad \forall M > \dfrac{\mathbb{E}X}{\delta}$

$\Rightarrow \mathbb{E}\left[X \mathbb{1}_{\{X_n>M\}}\right] < \varepsilon$. ⌋

A "uniformly integrable DCT"

- DCT says: If (X_n) is dominated ($|X_n| \le X$ with $\mathbb{E}X < \infty$) and $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{L^1} X$ ($\mathbb{E}|X_n - X| \to 0$)

- An extension, where domination $\mapsto$ U.I.:

THM (Uniformly Integrable DCT)
If (X_n) is uniformly integrable and $X_n \xrightarrow{a.s.} X_\infty$, then $X_n \xrightarrow{L^1} X_\infty \in L^1$

$\ulcorner$ U.I. criterion $\Rightarrow \sup_n \mathbb{E}|X_n| < \infty \overset{\text{Fatou}}{\Longrightarrow} \mathbb{E}|X_\infty| < \infty$.

Ex(a) & lem 1 $\Rightarrow Z_n := |X_n - X_\infty|$ is uniformly integrable and converges to 0 a.s. (by assumption). $\Rightarrow \forall \varepsilon > 0$:

$$\mathbb{E} Z_n = \underbrace{\mathbb{E}\left[Z_n \mathbb{1}_{\{Z_n > M\}}\right]}_{\text{fix } M \text{ s.t. } \le \varepsilon} + \underbrace{\mathbb{E}\left[Z_n \mathbb{1}_{\{Z_n \le M\}}\right]}_{\downarrow \text{a.s. by DCT}\ 0}$$

$\Rightarrow \limsup_n \mathbb{E} Z_n < \varepsilon \quad \forall \varepsilon > 0 \Rightarrow = 0.$ $\lrcorner$

THM (L¹ Martingale Convergence Thm)

Let (X_n) be a uniformly integrable martingale.

Then (X_n) converges a.s. and in L^1 to some r.v. $X_\infty \in L^1$.

Moreover, $$X_n = \mathbb{E}[X_\infty \mid \mathcal{F}_n] \quad \forall n. \qquad (*)$$

Thus, we can reconstruct the entire mgle (X_n) from its limit X_∞.

① Convergence:

- a.s. Ma Conv Thm $\Rightarrow$ a.s. convergence.
- U.I. DCT $\Rightarrow$ L^1 convergence.

② Identification of the limit

- By def. of conditional expectation, $(*)$ is equivalent to
$$\mathbb{E}(X_n \mathbb{1}_F) \overset{?}{=} \mathbb{E}[X_\infty \mathbb{1}_F] \quad \forall F \in \mathcal{F}_n \qquad (**)$$
- (X_n) is a mgle $\Rightarrow$ $X_n = \mathbb{E}[X_m \mid \mathcal{F}_n]$ $\forall m > n$.

$\Rightarrow$ by def. of conditional expectation, $\forall F \in \mathcal{F}_n$:
$$\mathbb{E}[X_n \mathbb{1}_F] = \mathbb{E}[X_m \mathbb{1}_F] \xrightarrow{m\to\infty} \mathbb{E}[X_\infty \mathbb{1}_F] \Rightarrow (**). \ \square$$

$|\text{Difference}| = |\mathbb{E}(X_m - X_\infty)\mathbb{1}_F| \le \mathbb{E}|X_m - X_\infty| \to 0$
by the L^1 convergence

Lévy's Upward & Downward Theorems.

THM (Lévy's Upward Thm) Let X be a r.v. with $\mathbb{E}|X| < \infty$ and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ be σ-algebras. Then

$$\mathbb{E}[X \mid \mathcal{F}_n] \longrightarrow \mathbb{E}[X \mid \mathcal{F}_\infty] \quad \text{a.s. and in } L^1$$

where $\mathcal{F}_\infty := \sigma(\bigcup_n \mathcal{F}_n)$, i.e. the smallest σ-algebra containing all $\mathcal{F}_n$.

Proof ① Convergence Lem 3 $\Rightarrow$

- $X_n := \mathbb{E}[X \mid \mathcal{F}_n]$ is a uniformly integrable martingale $\quad (*)$
- L^1 Mgle Conv Thm $\Longrightarrow$ $X_n \to X_\infty$ a.s. and in L^1.

  and $X_n = \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ $\forall n$. $\quad (**)$

② Identification of the limit: WTS

$$X_\infty \overset{?}{=} \mathbb{E}[X \mid \mathcal{F}_\infty]$$

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} X_\infty \text{ is } \mathcal{F}_\infty\text{-mble?} \; \left(\text{true since } X_n \xrightarrow{\text{a.s.}} X_\infty \;\&\; X_n \text{ is } \mathcal{F}_n \subset \mathcal{F}_\infty\text{-mble}\right) \\ \mathbb{E}[X_\infty \mathbb{1}_F] \overset{?}{=} \mathbb{E}[X \mathbb{1}_F] \quad \forall F \in \mathcal{F}_\infty \end{cases}$$

$(*)$ & $(**)$ $\Rightarrow$ $\mathbb{E}[X_\infty \mid \mathcal{F}_n] = \mathbb{E}[X \mid \mathcal{F}_n]$ $\forall n$

$\Rightarrow$ $\mathbb{E}[X_\infty \mathbb{1}_F] = \mathbb{E}[X \mathbb{1}_F]$ $\forall F \in \bigcup_n \mathcal{F}_n$ ← π-system

Uniqueness of measures (wlog $X \geq 0$) $\Rightarrow$ same holds $\forall F \in \sigma(\bigcup_n \mathcal{F}_n) = \mathcal{F}_\infty$. $\blacksquare$

THM (Lévy's downward thm) Let X be a r.v. with $\mathbb{E}|X|<\infty$ and $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$ be σ-algebras. Then

$$\mathbb{E}[X \mid \mathcal{F}_n] \longrightarrow \mathbb{E}[X \mid \mathcal{F}_\infty] \quad \text{a.s. and in } L^1$$

where $\mathcal{F}_\infty := \bigcap_n \mathcal{F}_n$.

① Convergence

$X_n := \mathbb{E}[X \mid \mathcal{F}_n]$ is a "reverse martingale". Lem $\Rightarrow (X_n)$ is unif. integrable.

$\forall N$, $(X_N, X_{N-1}, \dots, X_2, X_1)$ is a martingale.

$\Rightarrow$ the upcrossing ineq. holds for it.

Using it in the same way as in the proof of a.s. mgle conv. thm

$$\Rightarrow \qquad X_n \xrightarrow{a.s.} X_\infty \qquad \text{for some r.v. } X_\infty \in L^1 \quad \text{(DIY)}$$

$$\text{U.I. DCT} \Rightarrow \qquad X_n \xrightarrow{L^1} X_\infty$$

② Identification of the limit: WTS

$$X_\infty \overset{?}{=} \mathbb{E}[X \mid \mathcal{F}_\infty]$$

$$\Leftrightarrow \begin{cases} \text{(a) } X_\infty \text{ is } \mathcal{F}_\infty\text{-mble ?} \\ \text{(b) } \mathbb{E}[X_\infty \mathbb{1}_F] \overset{?}{=} \mathbb{E}[X \mathbb{1}_F] \quad \forall F \in \mathcal{F}_\infty \end{cases}$$

(a) true since $X_\infty \overset{a.s.}{=} \lim_m X_m$ & X_m is $\mathcal{F}_m \subset \mathcal{F}_n$-mble $\forall m > n$

$\Rightarrow X_\infty$ is $\mathcal{F}_n$-mble $\forall n$ $\Rightarrow$ $\mathcal{F}_\infty$-mble

(b) $X_n = \mathbb{E}[X \mid \mathcal{F}_n] \Rightarrow$

$$\mathbb{E}[X_n \mathbb{1}_F] = \mathbb{E}[X \mathbb{1}_F] \quad \forall F \in \mathcal{F}_\infty \subset \mathcal{F}_n$$

a.s. $\downarrow$ since $X_n \xrightarrow{L^1} X_\infty$

$\mathbb{E}[X_\infty \mathbb{1}_F]$