

## APPLICATION: 0-1 LAW

THM [Lévy's 0-1 Law] Let  $X_1, X_2, \dots$  be indep. r.v.'s and  $\tau := \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \dots)$  denote the tail  $\sigma$ -algebra. Then  $\forall \tau$ -measurable r.v.  $X$  is a.s. constant:  $X \stackrel{\text{a.s.}}{=} \mathbb{E}X$ .

$$\mathcal{F}_n := \sigma(X_1, \dots, X_n) \uparrow \sigma(X_1, X_2, \dots) =: \mathcal{F}_{\infty}$$

Lévy's upward thm  $\Rightarrow$

$$\mathbb{E}[X | \mathcal{F}_n] \xrightarrow{\text{a.s.}} \mathbb{E}[X | \mathcal{F}_{\infty}]$$

$$\left. \begin{array}{l} \parallel X \text{ is } \tau\text{-mble} \ \& \ \tau \perp \mathcal{F}_n \\ \mathbb{E}X \end{array} \right\} \parallel X \text{ is } \tau \subset \mathcal{F}_{\infty}\text{-mble} \\ \left. \begin{array}{l} X \\ \mathbb{E}X \end{array} \right\}$$

• Example:  $\forall$  event  $E \in \tau$ , consider  $X := \mathbb{1}_E$ .

Lévy 0-1  $\Rightarrow \mathbb{E}X \stackrel{\text{a.s.}}{=} X \Rightarrow \mathbb{P}(E) \in \{0, 1\}$ , i.e. Kolmogorov's 0-1 Law follows

**HW** vice versa, Kolm 0-1  $\Rightarrow$  Lévy 0-1 via HW (extremely opinionated)  
r.v.

## APPLICATION: SLLN

(let's give a ngle proof of (p.92))

Thm (Strong Law of Large Numbers) [Kolmogorov 1930]

Let  $X_1, X_2, \dots$  be iid r.v.'s with finite mean  $\mathbb{E}X_i = \mu$ .

Then  $S_n := X_1 + \dots + X_n$  satisfies

$$\frac{S_n}{n} \rightarrow \mu \text{ a.s. and in } L^1$$

- Kolmogorov's thm (proved earlier) does not need iid.

Q Does it hold for  $L^1$  convergence?

$$\mathcal{F}_n := \sigma(S_n, S_{n+1}, \dots) \downarrow \bigcap_{n=1}^{\infty} \mathcal{F}_n =: \mathcal{F}_{\infty}$$

Levy's downward thm  $\Rightarrow$

$$\mathbb{E}[X_1 | \mathcal{F}_n] \longrightarrow \mathbb{E}[X_1 | \mathcal{F}_{\infty}] \text{ a.s. and in } L^1$$

|| Ex. before

$S_n/n$

|| ? a.s.

$\mu$

• We know:

$$\mathbb{E}[X_1 | \mathcal{F}_{\infty}] = \lim_n \frac{S_n}{n} \in \mathcal{D}(X_1, X_2, \dots)$$

(changing finite # of  $X_i$ 's doesn't change  $\lim \frac{S_n}{n}$ )

$$\text{Levy's 0-1 Law } \Rightarrow \mathbb{E}[X_1 | \mathcal{F}_{\infty}] \stackrel{\text{a.s.}}{=} \mathbb{E}[\mathbb{E}[X_1 | \mathcal{F}_{\infty}]] = \mathbb{E}X_1 = \mu.$$

Application: A constructive approach to a generated  $\sigma$ -algebra  
(announced on p.16)

- Events  $E \in \sigma(E_1, \dots, E_n)$  look simple:  
 $E =$  finite Boolean formula  $(E_1, \dots, E_n)$  (e.g.  $E = E_1^c \cap E_2 \cap E_3^c \cap E_4$ )
- Events  $E \in \sigma(E_1, E_2, \dots)$  don't all look simple:  
 $E \neq$  countable Boolean formula  $(E_1, E_2, \dots)$  (e.g. for Borel,  $E = \mathbb{Q}$ -intervals)
- But:  $\uparrow$  true up to measure 0:

Thm (Approximating a generated  $\sigma$ -algebra)

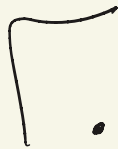
Let  $(\Omega, \mathcal{F}, P)$  be a prob. space,  $E_1, E_2, \dots \in \mathcal{F}$ .

Then  $\forall E \in \sigma(E_1, E_2, \dots) \exists A_n \in \sigma(E_1, \dots, E_n)$ :

$$\mathbb{1}_{A_n} \xrightarrow{\text{a.s.}} \mathbb{1}_E \quad \text{as } n \rightarrow \infty.$$



$$\begin{aligned} E' &= \limsup_n A_n \\ &= \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n \\ \Rightarrow P(E \Delta E') &= 0 \end{aligned}$$



•  $\mathcal{F}_n := \sigma(E_1, \dots, E_n), \quad X_n := E[\mathbb{1}_E | \mathcal{F}_n]$

Lévy's upward thm  $\Rightarrow$

$$X_n \xrightarrow{\text{a.s.}} E[\mathbb{1}_E | \mathcal{F}_{\infty}] = \mathbb{1}_E \quad \text{since } E \in \sigma\left(\bigcup_n \mathcal{F}_n\right) = \mathcal{F}_{\infty}.$$

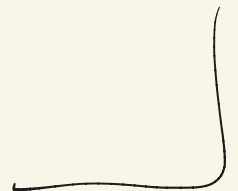
- "Quantize"  $X_n$  to 0/1 valued r.v. using  $f(x) := \begin{cases} 0, & x < 1/2 \\ 1, & x \geq 1/2 \end{cases}$

$f$  is continuous at 0 and 1  $\Rightarrow$

$$f(X_n) \xrightarrow{\text{a.s.}} f(\mathbb{1}_E) = \mathbb{1}_E$$

$\cap$   
 $\{0, 1\}$  and  $\mathcal{F}_n$ -measurable

$$\Rightarrow f(X_n) = \mathbb{1}_{A_n} \quad \text{for some } A_n \in \mathcal{F}_n.$$



# $L^p$ Martingale Convergence

## THM ( $L^p$ Martingale Convergence Thm)

let  $(X_n)$  be a martingale s.t.  $\sup_n \mathbb{E}|X_n|^p < \infty$  for some  $p > 1$ .

$X_n \rightarrow X_\infty$  a.s. and in  $L^p$

for some r.v.  $X_\infty \in L^p$ . Moreover,

$$X_n = \mathbb{E}[X_\infty | \mathcal{F}_n] \quad \forall n.$$

• A.s. MaConv Thm  $\Rightarrow X_n \xrightarrow{\text{a.s.}} X_\infty$  & "moreover" part.

•  $L^p$  convergence would follow by DCT if

$$Z := \sup_n |X_n| \in L^p$$

• To show this, we develop maximal ineq's for mgles.

Markov if  $k=1$

## THM (Doob's Maximal Inequality)

let  $(X_n)$  be a martingale. Then  $\forall n \in \mathbb{N}, \forall t > 0$ :

$$\mathbb{P} \left\{ \max_{1 \leq k \leq n} X_k \geq t \right\} \leq \frac{\mathbb{E} X_n^+}{t}$$

$$x^+ := \max(x, 0)$$

$\mathbb{E}$

CHECK WALSH p. 287

As in Etemadi's (p. 97), partition by the 1<sup>st</sup> exceedance:

$$E = \bigsqcup_{k=1}^n E_k \quad \text{where } E_k := \{X_1 < t, X_2 < t, \dots, X_{k-1} < t, X_k \geq t\}.$$

$$\Rightarrow t \cdot \mathbb{P}(E) = \sum_{k=1}^n t \cdot \mathbb{P}(E_k)$$

$$\bullet t \cdot \mathbb{P}(E_k) = \mathbb{E}[t \mathbb{1}_{E_k}] \leq \mathbb{E}[X_k \mathbb{1}_{E_k}] \stackrel{\text{mgle}}{=} \mathbb{E}[\mathbb{E}[X_n | \mathcal{F}_k] \mathbb{1}_{E_k}] \stackrel{\text{Tower}}{=} \mathbb{E}[X_n \mathbb{1}_{E_k}].$$

$\mathcal{F}_k$

$$\bullet \text{Substitute } \Rightarrow t \cdot \mathbb{P}(E) \leq \mathbb{E} \left[ \sum_{k=1}^n X_n \mathbb{1}_{E_k} \right] = \mathbb{E}[X_n \mathbb{1}_E] \leq \mathbb{E}[X_n^+].$$

• Remark The proof  $\Rightarrow$  MaMaxIneq is valid for submartingales:  
 | the highlighted equality becomes an inequality " $\leq$ ".

• Ex(a) If  $(X_n)$  is a mgle,  $|X_n|$  is a submartingale  $\Rightarrow$

$$P \left\{ \max_{1 \leq k \leq n} |X_k| \geq t \right\} \leq \frac{E|X_n|}{t} \quad (\text{Extension of Markov})$$

(b)  $(X_n^2)$  is also a submartingale  $\Rightarrow$

$$P \left\{ \max_{1 \leq k \leq n} |X_k| \geq t \right\} = P \left\{ \max_k X_k^2 \geq t^2 \right\} \leq \frac{E[X_n^2]}{t^2} \quad (\text{extension of Chebyshev})$$

$\neq$  an extension of Kolmogorov's max. ineq. for  $X_n = \text{sum of ind. r.v.s}$

Remark Stopping the proof 1 step early  $\Rightarrow \forall$  submartingale  $(X_n)$ ,  
 the event  $E := \left\{ \max_{1 \leq k \leq n} X_k \geq t \right\}$  satisfies  $P(E) \leq \frac{1}{t} E[X_n \mathbb{1}_E]$ .  
 We will use this in the proof of  $\searrow$

Thm ( $L^p$  martingale max. ineq.) let  $(X_n)$  be a martingale and  $p > 1$ . Then  $\forall n \in \mathbb{N}$ :

$$E \left[ \max_{1 \leq k \leq n} |X_k|^p \right] \leq \left( \frac{p}{p-1} \right)^p E|X_n|^p$$

In other words, **SKP**

$$M_n := \max_{1 \leq k \leq n} |X_k|$$

satisfies

$$\|M_n\|_{L^p} \leq p^* \|X_n\|_{L^p}$$

where  $\frac{1}{p} + \frac{1}{p^*} = 1$

Proof  $E M_n^p = \int_0^\infty P\{M_n^p \geq s\} ds$  (integrated tail formula)

$$= \int_0^\infty P t^{p-1} P\{M_n \geq t\} dt$$

Remark for submartingale  $|X_n|$   
 $\frac{1}{t} E[|X_n| \mathbb{1}_{\{M_n \geq t\}}]$

$$\leq E \left[ |X_n| \int_0^\infty P t^{p-2} \mathbb{1}_{\{M_n \geq t\}} dt \right]$$

$$= \frac{p}{p-1} \cdot E \left[ |X_n| M_n^{p-1} \right]$$

Hölder  $\leq \frac{p}{p-1} \cdot (E |X_n|^p)^{\frac{1}{p}} (E M_n^p)^{\frac{p-1}{p}}$

Rearranging & solving  $\Rightarrow (E M_n^p)^{\frac{1}{p}} \leq \frac{p}{p-1} (E |X_n|^p)^{\frac{1}{p}}$

Proof of  $L^p$  MaConv Thm

- Lem 2  $\Rightarrow (X_n)$  is uniformly integrable.
- $L^1$  MaConv Thm  $\Rightarrow X_n \xrightarrow{a.s.} X_\infty$ , and the "moreover" part holds.
- To upgrade to  $X_n \xrightarrow{L^p} X_\infty$ , use  $L^p$  Max Ineq & send  $n \rightarrow \infty$ .

$$\stackrel{MCT}{\Rightarrow} E \sup_n |X_n|^p \leq c_p \sup_n E |X_n|^p < \infty.$$

$$\cdot |X_n - X_\infty| \leq |X_n| + |X_\infty| \leq 2 \sup_n |X_n|$$

$$\Rightarrow |X_n - X_\infty|^p \leq 2^p \sup_n |X_n|^p \in L^1 \left. \vphantom{\sup_n |X_n|^p} \right\} \xrightarrow{D.C.T.} |X_n - X_\infty|^p \xrightarrow{L^1} 0.$$

Ex Deduce from  $L^p$  MaConv Thm a version of LNN for  $L^p$  convergence.

Remark  $L^p$  Max Inequality can't be extended to  $p=1$ :  
 otherwise the argument above would extend  $L^p$  MaConv Thm to  $p=1$  —  
 but we know it's false without a u.i. condition (see example in  $L^1$  MaConv Thm)

The most important case is  $p=2$ .

$$X_n = X_0 + \sum_{i=1}^n (X_i - X_{i-1}) \quad (*)$$

•  $X_0$  and all  $(X_i - X_{i-1})$  are orthogonal in  $L^2$  (see before)  $\Rightarrow$

$$\mathbb{E}X_n^2 = \mathbb{E}X_0^2 + \sum_{i=1}^n \mathbb{E}(X_i - X_{i-1})^2 \quad \xrightarrow{\text{MaCov Thm}}$$

### COR ( $L^2$ Martingale Convergence Thm)

Let  $(X_n)$  be a martingale with  $\mathbb{E}X_0^2 < \infty$  and

$$\sum_{i=1}^{\infty} \mathbb{E}(X_i - X_{i-1})^2 < \infty.$$

Then  $X_n \rightarrow X_{\infty}$  a.s. and in  $L^2$

for some r.v.  $X_{\infty} \in L^2$ . Moreover,

$$X_n = \mathbb{E}[X_{\infty} | \mathcal{F}_n] \quad \forall n.$$

• Direct proof of  $X_n \xrightarrow{L^2} X_{\infty}$ : (\*) & orthogonality  $\Rightarrow \forall m > n$ :

$$\|X_n - X_m\|_{L^2}^2 = \left\| \sum_{i=n+1}^m (X_i - X_{i-1}) \right\|_{L^2}^2 = \sum_{i=n+1}^m \|X_i - X_{i-1}\|_{L^2}^2$$

Assum  $\Rightarrow$  the series  $\sum_{i=1}^{\infty} \|X_i - X_{i-1}\|_{L^2}^2$  converges  $\Rightarrow$

$X_n - X_m$  is Cauchy in  $L^2 \xrightarrow{\text{completeness}} (X_n)$  converges in  $L^2$ .  $\square$ .

Remark (a) This generalizes Kolmogorov's 2 series thm.

(b) Vice versa, it can be proved in the same way as Kolm 2 series thm, but using Doob Max Ineq. instead of Kolmogorov's.  $\leftarrow$  HW

Add: Martingales in continuous time including def of quadratic variation