7 Sperner's Theorem

Def A family $\mathcal{F}$ of subsets of $\{1,\dots,n\}$ is called an antichain if no member of $\mathcal{F}$ is contained in another.

Ex: $\mathcal{F} = \{\text{all subsets of cardinality } k\}$

THM [Sperner] $\forall$ antichain satisfies $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Remark: equality achieved for $\mathcal{F} = \{\text{all subsets of cardinality } \lfloor n/2 \rfloor\}$.

Proof Let $\sigma \sim \mathrm{Unif}(S_n)$ be a random permutation. Consider a random chain

$$C_\sigma := \{\sigma(\{1,\dots,i\}) : i \in \{1,\dots,n\}\}$$

e.g. 3 6 1 2 5 4

$\forall$ antichain $\mathcal{F}$ can contain at most 1 set from any given chain $\Rightarrow$

$$|\mathcal{F} \cap C_\sigma| \leq 1.$$

On the other hand,

$$X := |\mathcal{F} \cap C_\sigma| = \sum_{F \in \mathcal{F}} \mathbb{1}_{\{F \in C_\sigma\}}$$

$$\Rightarrow \quad 1 \geq \mathbb{E}X = \sum_{F \in \mathcal{F}} \mathbb{P}\{F \in C_\sigma\} = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \geq |\mathcal{F}| \binom{n}{\lfloor n/2 \rfloor}. \qquad \square$$

C_σ contains exactly 1 set of size $|F|$, and this set is uniformly distributed among all $|F|$-element subsets of $\{1,\dots,n\}$

$\binom{n}{\lfloor n/2 \rfloor}$: the middle binomial coefficient is the largest.

8 Littlewood–Offord Inequality

- If $\pm$ signs are chosen at random, independently,

$$P\{\underbrace{\pm1\pm1\cdots\pm1}_{n\text{ (even)}}=0\}=P\{\#\text{pluses}=\tfrac{n}{2}\}=2^{-n}\binom{n}{n/2}\sim\frac{1}{\sqrt{n}}.$$

- If 1's are replaced with $\forall$ nonzero weights, even better:

THM [Littlewood–Offord; Erdös]

Let $a_1,\dots,a_n$ be nonzero numbers,
$\varepsilon_1,\dots,\varepsilon_n$ be independent Rademacher random var's.

Then $$P\left\{\sum_1^n \varepsilon_i a_i=0\right\}\le P\left\{\sum_1^n\varepsilon_i=0\right\}=2^{-n}\binom{n}{\lfloor n/2\rfloor}=\sqrt{\frac{2}{\pi n}}\,(1+o(1))$$

Proof
- WLOG $a_i>0$.
- WTS: there are $\le\binom{n}{\lfloor n/2\rfloor}$ choices of signs (ε_i) such that $\sum_1^n\varepsilon_i a_i=0$ $\quad(*)$
- Each such choice is uniquely determined by the set
$$F=\{i:\varepsilon_i=+1\}\subset\{1,\dots,n\}.$$
- These sets form an *antichain*

(Indeed, if $F\subset F'$, then one can change some "$-$" signs to "+" signs while maintaining $\sum_1^n\varepsilon_i a_i=0$

F: $+++-----$
F': $+++++---$

But this is impossible: since all $a_i>0$, the sum must get bigger.)

- Hence, by Sperner's theorem, $\exists$ at most $\binom{n}{\lfloor n/2\rfloor}$ such sets F.

$\Rightarrow(*)$ holds. $\square$

APPLICATION: HARDY–LITTLEWOOD MAX. INEQUALITY

Def Given a function $f:\mathbb{R}\to\mathbb{R}$ & a prob. measure μ on $\mathbb{R}$, the H-L max function is

$$(Mf)(x) := \sup_{I\ni x} \frac{1}{|I|}\int_I |f|\,d\mu \qquad (*)$$

over all bounded intervals $I\subset\mathbb{R}$ that contain x.

Thm (HL Max Ineq) $\forall\, p>1$, $\forall\, f\in L^p(\mathbb{R},\mu)$:

$$\|Mf\|_{L^p} \le C_p \|f\|_{L^p}$$

$\int_I$

WLOG $f\ge 0$. WLOG $|I|\le 1$ (otherwise $Mf \le \int_{\mathbb{R}} f\,d\mu \le \|f\|_{L^p}$ pointwise)

① For dyadic intervals: Let's first allow only dyadic intervals in $(*)$ — those of the form

$$[k2^{-n},(k+1)2^{-n}], \qquad k\in\mathbb{Z},\ n\in\{0,1,2,\dots\}$$

Call the corresponding dyadic max. function $\tilde{M}f$.

- Filtration: $\mathcal{F}_n := \sigma(\text{dyadic intervals of length } 2^{-n})$. Martingale:

$$X_n(x) := \mathbb{E}[f\mid\mathcal{F}_n](x) = \frac{1}{|I|}\int_I f\,d\mu$$ where $I\in\mathcal{F}_n$ is the unique interval $\ni x$.

- $\tilde{M}f = \max_n X_n$. L^p Ma Max Ineq $\Rightarrow$

$$\|\tilde{M}f\|_{L^p} \le C_p \|f\|_{L^p}.$$

② From dyadic to general intervals:

Fact: Let $I\subset\mathbb{R}$ be $\forall$ interval of length ≤ 1.
Then $\exists$ an interval J which is either a dyadic interval or a dyadic interval shifted by $1/3$ s.t.

$$I\subset J, \qquad |J|\le |I|.$$

Apply ① to the dyadic intervals & the shifted dyadic intervals,

sum up $\Rightarrow$ QED

Hw Extend to $\forall$ meas μ on $\mathbb{R}$ (not just prob. meas)

Application: A constructive approach to a generated σ-algebra

(announced on p.16)

- Events $E \in \sigma(E_1, \dots, E_n)$ look simple:

  E = finite Boolean formula $(E_1, \dots, E_n)$ (e.g. $E = E_1^c \cap E_2 \cap E_3^c \cap E_4$)

- Events $E \in \sigma(E_1, E_2, \dots)$ don't all look simple:

  $E \neq$ countable Boolean formula $(E_1, E_2, \dots)$ (e.g. for Borel, E_i = $\mathbb{Q}$-intervals)

- But: true up to null sets:

THM (Approximating a generated σ-algebra)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob. space, $E_1, E_2, \dots \in \mathcal{F}$.

Then $\forall\, E \in \sigma(E_1, E_2, \dots)$ $\exists$ $A_n \in \sigma(E_1, \dots, E_n)$:

$$\mathbb{1}_{A_n} \xrightarrow{\text{a.s.}} \mathbb{1}_E \quad \text{as } n \to \infty.$$

$E' := \limsup_n A_n$

$= \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$

$\Rightarrow \mathbb{P}(E \triangle E') = 0$

"countable Boolean formula" up to null sets

- $\mathcal{F}_n := \sigma(E_1, \dots, E_n)$, $X_n := \mathbb{E}[\mathbb{1}_E \mid \mathcal{F}_n]$

  Lévy's upward thm $\Rightarrow$

  $$X_n \xrightarrow{\text{a.s.}} \mathbb{E}[\mathbb{1}_E \mid \mathcal{F}_\infty] = \mathbb{1}_E \quad \text{since } E \overset{\text{asm}}{\in} \sigma\Big(\bigcup_n \mathcal{F}_n\Big) = \mathcal{F}_\infty.$$

- "Quantize" X_n to 0/1 valued r.v. using $f(x) := \begin{cases} 0, & x < 1/2 \\ 1, & x \geq 1/2 \end{cases}$

  f is continuous at 0 and 1 $\Rightarrow$

  $$\underbrace{f(X_n)}_{\in\, \{0,1\} \text{ and } \mathcal{F}_n\text{-measurable}} \xrightarrow{\text{a.s.}} f(\mathbb{1}_E) = \mathbb{1}_E$$

  $\Rightarrow f(X_n) = \mathbb{1}_{A_n}$ for some $A_n \in \mathcal{F}_n$.

APPLICATION: A Probabilistic Proof of Stirling's Approximation

Thm (Stirling's Approximation) $\quad n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1+o(1))$ as $n\to\infty$

Proof following [Nils Lid Hjort, Emil Aas Stoltenberg, Probability Proofs of Stirling '2024]

- $Y_n \sim \text{Poisson}(n)$ can be expressed as a sum of n iid Poisson(1) r.v's; each has mean 1, variance 1, third moment $\le C$.

- Apply Wasserstein CLT for
$$W := \frac{Y_n - n}{\sqrt{n}} \implies W_1(W, Z) \le \frac{C}{\sqrt{n}} \to 0 \qquad \text{where } Z \sim N(0,1)$$

- The function $h(x) = \max(x,0)$ is 1-Lipschitz $\implies$
$$|\mathbb{E}h(W) - \mathbb{E}h(Z)| \to 0 \quad \text{as } n\to\infty.$$

- $$\mathbb{E}h(Z) = \frac{1}{\sqrt{2\pi}}\int_0^\infty x e^{-x^2/2}\,dx \overset{y=x^2/2}{=\!=} \frac{1}{\sqrt{2\pi}}.$$

- $$\mathbb{E}h(W) = \sum_{k=n}^\infty \frac{k-n}{\sqrt{n}} \overset{e^{-n}n^k/k!}{\overset{\shortparallel}{p(k)}}$$
$$= \frac{1}{\sqrt{n}}\Big(np(n) - \underbrace{np(n) + (n+1)p(n+1)}_{=\ \text{check!}} \underbrace{- np(n+1) + (n+2)p(n+2)}_{=\ \text{check!}} \underbrace{- np(n+2) + \cdots}_{=}\Big)$$
$$= \frac{np(n)}{\sqrt{n}} = \frac{\sqrt{n}e^{-n}n^n}{n!}.$$

- Hence $\dfrac{\sqrt{n}e^{-n}n^n}{n!} \to \dfrac{1}{\sqrt{2\pi}}.$ $\quad\square$