Last Episode: <u>factorial</u> $n! = 1 \cdot 2 \cdot 3 \cdots n$.

$0! = 0, \quad 1! = 1, \quad 2! = 1 \cdot 2 = 2, \quad 3! = 1 \cdot 2 \cdot 3 = 6, \ldots$

BIRTHDAY PROBLEM 2: What is the probability that among your 100 friends, some have the same BD?

As before:

. Complementary problem: Prob (all BDs are different) = ?

$$\#(\text{ways to assign BD's}) = \underbrace{365 \cdots 365}_{100} = 365^{100}$$

- All assignments are equally likely.

$$\#\left(\begin{array}{l}\text{ways to assign BD's}\\ \text{so that all are different}\end{array}\right) = \underbrace{365 \cdot 364 \cdot 363 \cdots 266}_{100}$$

$$= \frac{365 \cdot 364 \cdots 266 \cdot 265 \cdots 3 \cdot 2 \cdot 1}{265 \cdots 3 \cdot 2 \cdot 1} = \frac{365!}{265!}$$

$$\Rightarrow \text{Prob (all BD's are different)} = \frac{365!/265!}{365^{100}} \approx 3 \cdot 10^{-7} \quad (*)$$

$$\Rightarrow \text{Prob. (some BD's are the same)} = 1 - (*) = 0.\underbrace{999999}_{6}7$$

- **Problem**: Computer can't calculate $(*)$: $365! \approx \infty$
- $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$ grows too fast

  e.g. $10! = 3{,}628{,}800$

- **Stirling's approximation**:

$$\boxed{n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \quad \text{as } n \to \infty.$$

- For $n = 365$, $k = 100$,

$$(*) = \frac{n!}{(n-k)!\, n^k} \approx \frac{\sqrt{2\pi n}\, n^n}{e^n} \cdot \frac{e^{n-k}}{\sqrt{2\pi(n-k)}\,(n-k)^{n-k}} \cdot \frac{1}{n^k}$$

$$\underset{\text{DIY}}{\overset{\text{simplify}}{=}} \boxed{\left(\frac{n}{n-k}\right)^{n-k+\frac{1}{2}} e^{-k}} \leftarrow \text{computable, gives } (*)$$