S1: E5

- last episode:

$$\#\left(\text{ways to choose } \overset{\text{unordered set}}{k} \text{ from } n \text{ objects}\right) = \frac{n!}{k!(n-k)!} =: \overset{\text{binomial coefficient}}{\binom{n}{k}}$$

- $\binom{n}{n} = \frac{n!}{n!\,0!} = 1$. $\binom{n}{0} = \frac{n!}{0!\,n!} = 1$.

invite all friends   invite no friends

This is why we set $\boxed{0! = 1.}$

Ex An airline operates 7 daily flights NY → LA

Each flight can be late or on time, e.g.

LTLTTLT

late on time

(a) How many scenarios are possible in which 3 flights are late & 4 are on time?

$$\binom{7}{3} = \binom{7}{4} = 35$$

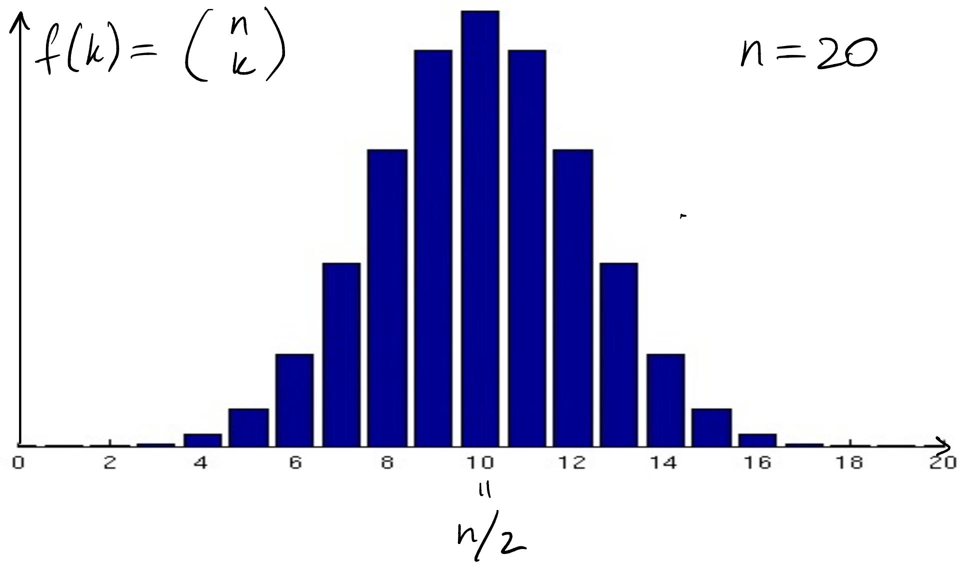
choose which flights are late

equivalently, choose which flights are on time

More generally,

$$\boxed{\underline{\text{Prop}} \text{ (Symmetry)} \quad \binom{n}{k} = \binom{n}{n-k}}$$

i.e. the binomial coefficients are symmetric about $n/2$:



(6) What if we require that no late flights are consequtive?

(i.e. LTLLTTT is not allowed)

Line up the 4 on-time flights with $\underline{\text{spaces}}$ next to them:

$$\_\, T \,\_\, T \,\_\, T \,\_\, T \,\_$$

Each space has room for ≤ 1 late flight.

$\Rightarrow$ #(scenarios) = #(ways to choose 3 from 5 spaces)

$$= \binom{5}{3} = 10.$$

Ex (a) How many natural solutions does the equation i.e. $x, y \in \mathbb{N} = \{1, 2, 3, \dots\}$

$x + y = 5$ have?

$$\left.\begin{aligned} 1+4&=5 \\ 2+3&=5 \\ 3+2&=5 \\ 4+1&=5 \end{aligned}\right\} \Rightarrow 4 \text{ solutions}$$

(b) What about $x + y + z = 6$? e.g. $2+3+1 = 6$, $1+4+1 = 6$ ...

$$\underbrace{1+1}_{x=2} \oplus \underbrace{1+1+1}_{y=3} \oplus \underbrace{1}_{z=1} = 6$$

$\#(\text{sols}) = \#(\text{ways to choose 2 from 5 "+"s}) = \binom{5}{2} = 10$

Generally:

$$\boxed{\textbf{THM} \text{ The equation } x_1 + x_2 + \dots + x_k = n \text{ has } \binom{n-1}{k-1} \text{ natural solutions.}}$$

Remark: if we want to count $2+3+1=6$ and $3+2+1=6$ as the same solution, $\#(\text{sols}) = \#$"partitions" of n

No closed-form expression; see Young diagrams.