$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

...

5 factors

$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$

$= 1 \cdot y^5 + \binom{5}{1} x^1 y^4 + \binom{5}{2} x^2 y^3 + \binom{5}{3} x^3 y^2 + \binom{5}{4} x^4 y^1 + 1 \cdot x^5$

choose 1 factor for x & the other 4 for y ... choose 3 factors for x the other 2 for y ...

$= y^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5xy^4 + x^5$.

Generally:

⇓

BINOMIAL THEOREM $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Letting $x = y = 1$ in B.T, we get

COROLLARY $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ Hard to prove directly

Alternative proof: both sides = #ways to choose any subset of n friends for a party

Another identity:

Pascal's Rule $\forall n, k$: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Prove directly (DIY)

Combinatorial proof:

$\overbrace{\text{Leo} \quad \underbrace{\text{Sandy} \quad \text{Xu} \quad \text{Isabel} \ldots}_{n-1}}^{n \text{ friends}}$

$\#(\text{ways to invite } k \text{ friends}) = \binom{n}{k}$

$\|$

$\#(\text{ways to invite } k \text{ friends NOT including Leo}) + \#(\cdots \text{including Leo})$

Leo is excluded

Leo is already included

$= \binom{n-1}{k} + \binom{n-1}{k-1}$

invite k-1 more

$\square$

P.R yields a recursion to compute binomial coeff's:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$\binom{3}{k}$

$\binom{4}{k}$

$\binom{5}{k}$

Thus $(x+y)^5 = y^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5xy^4 + x^5$.

$(x+y)^6 = \cdots$ (DIY)