THE AXIOMS OF PROBABILITY

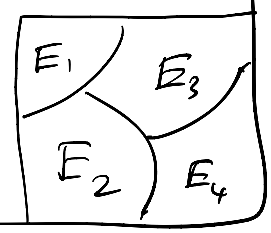
Def Let S be a sample space.

A probability is an assignment of a number $P(E)$ to each event $E \subset S$, which satisfies the following properties ("axioms"):

1. $P(E) \geq 0 \quad \forall E \subset S$
2. $P(S) = 1$
3. $\forall$ mutually exclusive events $E_1, E_2, \ldots, E_n$ (where n may be finite or ∞),

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

"Additivity"



Remarks 1. If $n = \infty$, this series is required to converge

2. Probability is a function $P: 2^S \to [0,1]$
   power set

3. Other functions that satisfy axioms 1 & 2:
   length, area, volume, mass; "Measure"

   percentage satisfies all axioms.

UNIFORM PROBABILITY

Def Let S be a finite set.
If all outcomes are equally likely, ← i.e. $P(\{s\}) = P(\{t\})$ $\forall s, t \in S$
P is called the <u>uniform probability</u> on S.

Proposition In this case,

$$P(E) = \frac{|E|}{|S|} \qquad \forall \text{ event } E \subset S$$

Proof.

- P is uniform $\Rightarrow$ all outcomes must have prob. $= \frac{1}{|S|}$

(Indeed, $\underset{\uparrow \text{ axiom 2}}{1 = P(S)} = P\left(\bigcup_{s \in S} \{s\}\right) \underset{\uparrow \text{ axiom 3}}{=} \sum_{s \in S} \underbrace{P(\{s\})}_{\uparrow \text{ all equal}} = N \cdot P(\{s\})$)

- $\forall$ event $E \subset S$

$$P(E) = P\left(\bigcup_{s \in E} \{s\}\right) = \underbrace{\sum_{s \in E}}_{|E| \text{ terms}} \underbrace{P(\{s\})}_{\text{each} = \frac{1}{|S|}} = \frac{|E|}{|S|} \qquad \text{QED.}$$

__Ex__ (a) Flip 2 coins, $S = \{HH, HT, TH, TT\}$

all outcomes are equally likely

$\Rightarrow P(\text{one head}) = P(\{HT, TH\}) = \frac{2}{4} = \frac{1}{2}$

(b) Choose 10 rats from 100 (= 40 sick, 60 healthy)

$S = \{\text{all subsets of 10 rats}\}$

All selections are equally likely $\Rightarrow$

for $E = \{4 \text{ sick}, 6 \text{ healthy}\}$

$$P(E) = \frac{|E|}{|S|} = \frac{\binom{40}{4}\binom{60}{6}}{\binom{100}{10}} \approx 0.26$$

__Note__: not all probabilities are uniform, e.g.

Experiment = renting a Netflix series

$S = \{\text{all Netflix series}\}$

P is NOT uniform, due to variation of preferences.