The Axioms of Probability

**Definition**
Let $S$ be a sample space.

A probability is an assignment of a number $P(E)$ to each event $E \subseteq S$, which satisfies the following properties ("axioms"):

1. $P(E) \geq 0 \quad \forall E \subseteq S$
2. $P(S) = 1$
3. For mutually exclusive events $E_1, E_2, \ldots, E_n$ (where $n$ may be finite or $\infty$),
   
   \[ P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) \]

   "Additivity"

**Remarks**
1. If $n = \infty$, this series is required to converge.
2. Probability is a function $P : 2^S \rightarrow [0,1]$
3. Other functions that satisfy axioms 1 & 2:
   - Length, area, volume, mass; "Measure"
   - Percentage satisfies all axioms.
UNIFORM PROBABILITY

**Def** Let $S$ be a finite set.
If all outcomes are equally likely,
$P$ is called the **uniform probability** on $S$.

**Proposition** In this case,

$$P(E) = \frac{|E|}{|S|} \quad \forall \text{ event } E \subset S$$

**Proof.**

- $P$ is uniform $\Rightarrow$ all outcomes must have prob. $= \frac{1}{|S|}$

\[ \text{Indeed, } 1 = P(S) = P(\bigcup_{s \in S} \{s\}) = \sum_{s \in S} P(\{s\}) = N \cdot P(\{s\}) \]

- $\forall \text{ event } E \subset S$

$$P(E) = P(\bigcup_{s \in E} \{s\}) = \sum_{s \in E} P(\{s\}) = \frac{|E|}{|S|} \quad \text{ QED.}$$
(a) Flip 2 coins, \( S = \{ HK, KT, TH, TT \} \)

all outcomes are equally likely

\[ P(\text{one head}) = P(\{KT, TH\}) = \frac{2}{4} = \frac{1}{2} \]

(b) Choose 10 rats from 100 (=40 sick, 60 healthy)

\( S = \{ \text{all subsets of 10 rats} \} \)

All selections are equally likely \( \rightarrow \)

for \( E = 4 \text{ sick} \& 6 \text{ healthy} \)

\[ P(E) = \frac{\binom{40}{4} \binom{60}{6}}{\binom{100}{10}} = 0.26 \]

**Note:** not all probabilities are uniform, e.g.

Experiment = renting a Netflix series

\( S = \{ \text{all Netflix series} \} \)

\( P \) is not uniform, due to variation of preferences.