S2 : E5

Prop. 4.3 (Inclusion-Exclusion Principle) $\forall$ events E, F:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

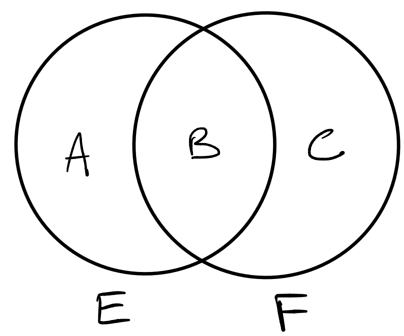
$\uparrow$ LHS $\qquad\qquad$ $\uparrow$ RHS

$A := E \setminus F$, $B := E \cap F$, $C := F \setminus E$

Then A, B, C are mutually exclusive and

$E = A \cup B$, $\quad F = B \cup C$, $\quad E \cup F = A \cup B \cup C$

(check!)



$\Rightarrow$ LHS $= P(A) + P(B) + P(C)$

RHS $= \big(P(A) + P(\cancel{B})\big) + \big(P(B) + P(C)\big) - P(\cancel{B}) = P(A) + P(B) + P(C)$.

Same.

Ex A construction company says that 97% of its homes are earthquake-safe (E), 96% are flood-safe (F), and 91% are both ($E \cap F$).

Disprove their claim.

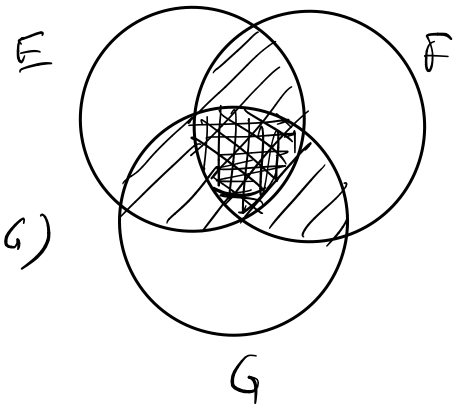
Experiment: Choose a home at random, with uniform probability.

IEP $\Rightarrow$

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(E \cap F) \\ &= 0{,}97 + 0{,}96 - 0{,}94 \\ &= 1{,}02 > 1 \ \lightning \text{ axiom } 1 \end{aligned}$$

More generally:

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G),$$ etc.



Prop. 4.4 (General IEP)

$\forall$ events $E_1, \ldots, E_N$:

$$P(E_1 \cup \cdots \cup E_N) = \underbrace{\sum_{i=1}^{N} P(E_i)}_{N \text{ terms}} - \underbrace{\sum_{\substack{i,j \\ \text{distinct}}} P(E_i \cap E_j)}_{\binom{N}{2} \text{ terms}} + \underbrace{\sum_{\substack{i,j,k \\ \text{distinct}}} P(E_i \cap E_j \cap E_k)}_{\binom{N}{3} \text{ terms}}$$

$$+ (-1)^{N+1} P(E_1 \cap \cdots \cap E_N).$$

Ex (The matching problem)

N exams are returned to N students at random.
Find the probability of a "derangement" — the event where NO students get their own exam

- Experiment: return N exams to N students.
  $S = \{\text{all ways to return } N \text{ exams}\}$, $|S| = N!$
- Complement: $E :=$ "at least one student gets own exam"

$$E = \bigcup_{i=1}^{N} E_i, \text{ where } E_i = \text{"student } i \text{ gets own exam"}$$

Use IEP:

(1) $P(E_i) = \frac{|E_i|}{|S|} = \frac{(N-1)!}{N!} = \frac{1}{N}$ ← # of ways to return N-1 exams (after student i gets their own)

(2) $P(E_i \cap E_j) = P\{$ both students i,j get their own exams $\}$

$= \frac{|E_i \cap E_j|}{|S|} = \frac{(N-2)!}{N!}$ ← # ways to return N-2 exams (after studs i,j get their own)

(3) $P(E_i \cap E_j \cap E_k) = \frac{(N-3)!}{N!}$

. . . IEP ⇒

$$P(E) = N \cdot \frac{1}{N} - \underbrace{\binom{N}{2}}_{= \frac{N!}{2!(N-2)!}} \frac{(N-2)!}{N!} + \underbrace{\binom{N}{3}}_{= \frac{N!}{3!(N-3)!}} \frac{(N-3)!}{N!} - \dots + (-1)^{N+1} \binom{N}{N} \frac{(N-N)!}{N!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{N} \frac{1}{N!}$$

$$\Rightarrow P(E^c) = 1 - P(E) = \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^N}{N!} \simeq \boxed{\frac{1}{e}} \approx 0.37.$$

(recall $e^x = \sum_{n\geq 0}^{\infty} \frac{x^n}{n!}$ $\quad e^{-1} = \sum_{n \geq 0}^{\infty} \frac{(-1)^n}{n!}$)