S3:E10

Secretary problem, a.k.a. Best prize problem

- We are presented with n prizes, in sequence.
- Upon seeing a prize, we must accept it (and end the game) or reject it (and move to the next prize). No going back.
- The only info we have at $\forall$ time is how the current prize compares to the prizes already seen.
- We want to $\underbrace{\boxed{\text{pick the best prize.}}}_{E}$ What shall we do?

- Strategy: reject the first k prizes; accept the first one that is better than those k.

Let's compute P(E) and optimize k.

- Condition on the position of best prize:

$$\boxed{B_i = \text{"i-th prize is the best"}.}$$

LTP $\Rightarrow$

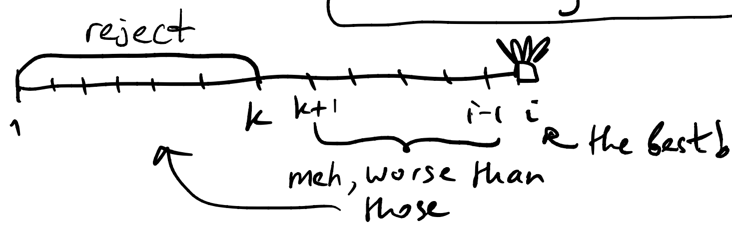
$$P(E) = \sum_{i=1}^{n} \underbrace{P(E|B_i)}_{= \, ?} \underbrace{P(B_i)}_{= \frac{1}{n}}$$

(all are equally likely)

- $\forall\, i \le k$ $\quad P(E|B_i) = 0$ (we reject the first k prizes)
- $\forall\, i > k$: assume B_i occurs, i.e. i-th prize is the best.

  We pick it iff all prizes $k+1, \dots, i-1$ are <u>worse</u> than the first k. (Otherwise we lose it)

  Thus, E occurs iff $\boxed{\text{the best among the } i-1 \text{ prizes is among the first } k.}$



- The latter happens with prob. $\dfrac{k}{i-1}$

  ($\forall$ i-1 positions equally likely)

$$\Rightarrow \quad P(E|B_i) = \frac{k}{i-1}$$

$$\Rightarrow P(E) = \sum_{i=k+1}^{n} \frac{k}{i-1} \cdot \frac{1}{n} = \boxed{\frac{1}{n} \sum_{i=k}^{n-1} \frac{k}{i}}$$

$$\approx \frac{k}{n} \int_{k}^{n-1} \frac{dx}{x} = \frac{k}{n} \ln\left(\frac{n-1}{k}\right) \approx \boxed{\frac{k}{n} \ln \frac{n}{k}} = -x \ln x$$

where $x = k/n$

Maximize $\Rightarrow$ $\boxed{x = 1/e \,, \quad P(E) = 1/e}$

<u>Ans</u>: reject the first $\frac{n}{e}$ prizes, pick the first one better than these.

Probability to pick the best is $\approx 1/e \approx 0.37$

$\approx$ regardless of n!!

<u>Remark</u>: Optimal stopping.