Secretary problem, a.k.a. Best prize problem

- We are presented with n prizes, in sequence.
- Upon seeing a prize, we must accept it (and end the game) or reject it (and move to the next prize). No going back.
- The only info we have at any time is how the current prize compares to the prizes already seen.
- We want to pick the best prize. What shall we do?

Strategy:
- reject the first k prizes;
- accept the first one that is better than those k.

Let's compute $P(E)$ and optimize $k$.

Condition on the position of best prize:

\[ B_i = \text{"i-th prize is the best".} \]

**LTP:**

\[
P(E) = \sum_{i=1}^{n} P(E | B_i) P(B_i) \]

- \[
\frac{1}{n} \quad \text{(all are equally likely)}
\]
\[ \forall i \leq k \quad P(EB_i) = 0 \quad \text{(we reject the first k prizes)} \]

\[ \forall i > k : \ \text{assume } B_i \text{ occurs, i.e. } i \text{-th prize is the best.} \]

We pick it iff all prizes \( k+1, \ldots, i-1 \) are worse than the first \( k \). (Otherwise we lose it.)

Thus, \( E \) occurs iff the best among the \( i-1 \) prizes is among the first \( k \).

- The latter happens with prob. \( \frac{k}{i-1} \)

\[ \Rightarrow \quad P(EB_i) = \frac{k}{i-1} \]

\[ \Rightarrow \quad P(E) = \sum_{i=k+1}^{n} \frac{k}{i-1} \cdot \frac{1}{n} = \frac{1}{n} \sum_{i=k}^{n-1} \frac{k}{i} = \frac{k}{n} \ln \left( \frac{n-1}{k} \right) = - \frac{k}{n} \ln \frac{n}{k} \]

Maximize \( \Rightarrow \quad x = \frac{1}{e}, \quad P(E) = \frac{1}{e} \)

\[ \text{Ans: reject the first } \frac{n}{e} \text{ prizes, pick the first one better than these.} \]

Probability to pick the best is \( \approx \frac{1}{e} \approx 0.37 \)

\[ \approx \text{regardless of } n \]

Remark: Optimal stopping.