Independent events

- Ex: owning a car $\perp\!\!\!\perp$ owning a cat ✓
  owning a car $\not\!\perp\!\!\!\perp$ getting into a traffic accident ✗
- Intuitively, E and F are independent if
  $$P(E|F) = P(E), \qquad P(F|E) = P(F) \qquad (*)$$
  i.e. the occurrence of F does not affect the likelihood of E, and vice versa.
- Ex $\quad P\{\underbrace{10\text{'th flip} = H}_{E} \mid \underbrace{\text{first 9 flips} = H}_{F}\} = P(E) = \frac{1}{2} \qquad (E \perp\!\!\!\perp F)$
- Rewrite $(*)$: $\quad \frac{P(E\cap F)}{P(F)} = P(E), \qquad \frac{P(F\cap E)}{P(E)} = P(F)$
  $$\Updownarrow \qquad\qquad \Updownarrow$$
  $$P(E\cap F) = P(E)\cdot P(F).$$

Def Events E, F are <u>independent</u> $(E \perp\!\!\!\perp F)$ if
$$P(E\cap F) = P(E)\cdot P(F).$$

- In Example above, $P\{\text{both flips} = H\} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} \Rightarrow$ indep.
- <u>Warning</u>: independent $\neq$ mutually exclusive!
  In fact, mutually exclusive sets are typically dependent
  Example: R_1 = "finding a missing hiker in Region 1"
  R_2 = " ——————— Region 2"
  are mutually exclusive, very dependent: $P(R_1 | R_2) = 0$

Independence of 3 events:

Def E, F, G are (jointly) independent if

$P(E\cap F)=P(E)P(F)$; $P(E\cap G)=P(E)P(G)$; $P(F\cap G)=P(F)P(G)$.

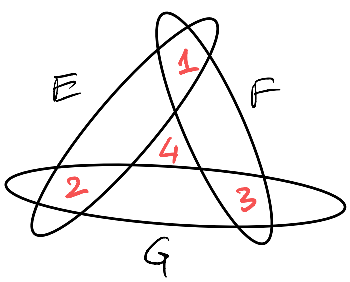
"pairwise independent"

$P(E\cap F\cap G)=P(E)\,P(F)\,P(G)$.

Ex: Pairwise independence $\not\Rightarrow$ independence:

Experiment: pick a number from $\underbrace{\{1,2,3,4\}}_{=S}$ at random and with uniform probability. Events:

$E :=$ "1 or 2", $F =$ "1 or 3", $G =$ "2 or 3"



$P(E)=P(F)=P(G)=\frac{2}{4}=\frac{1}{2}$;

$P(\underbrace{E\cap F}_{=\{1\}})=\frac{1}{4}=P(E)\,P(F) \Rightarrow E \perp F$.

Similarly for the other pairs $\Rightarrow$ $\boxed{\text{pairwise indep.}}$

- But $P(\underbrace{E\cap F\cap G}_{=\emptyset})=0 \neq P(E)\,P(F)\,P(G) \Rightarrow$ $\boxed{\text{NOT indep}}$

For $\forall$ number of events:

Def $E_1, E_2, \ldots$ are (jointly) independent if

$$P\Big(\bigcap_{i\in I} E_i\Big)=\prod_{i\in I} P(E_i) \quad \forall \text{ subset } I.$$

Prop (Independence is stable)

If $E_1, E_2, \dots, F_1, F_2, \dots$ are all independent, then

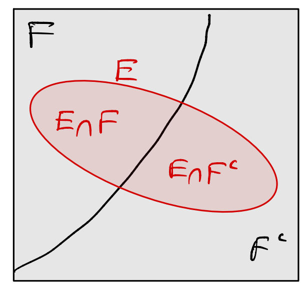
$\forall$ event formed of $E_1, E_2, \dots$ is independent of

$\forall$ event formed of $F_1, F_2, \dots$

using $\cap$, $\cup$, complements

First, check a couple of special cases:

if E, F, G are indep. then:



(a) Complements: $E \perp\!\!\!\perp F^c$

mutually exclusive

$E = (E\cap F) \cup (E \cap F^c) \Rightarrow$

$P(E) = \underbrace{P(E\cap F)}_{\substack{\| \\ P(E)\cdot P(F) \text{ by independence}}} + P(E\cap F^c) \Rightarrow P(E\cap F^c) = P(E) - P(E)\cdot P(F)$

LTP

$= P(E)\left(1 - P(F)\right) = P(E)\,P(F^c).$

(b) Intersections: $E \perp\!\!\!\perp (F\cap G)$?

$P(E\cap(F\cap G)) = P(E\cap F\cap G) = P(E)\,P(F)\,P(G)$ (indep)

$\underset{\uparrow F \perp\!\!\!\perp G}{=} P(E)\,P(F\cap G)$

(c) $\forall$ other set operations can be built from complements & inters's:

union $\quad F\cup G = (F^c \cap G^c)^c \quad$ (de Morgan)

iterate.