**Independent events**

- **Ex:** owning a car \( \perp \) owning a cat \( \checkmark \)
  owning a car \( \not\perp \) getting into a traffic accident \( \times \)

- Intuitively, \( E \) and \( F \) are independent if
  \[
  P(E|F) = P(E), \quad P(F|E) = P(F) \quad (\times)
  \]
  i.e. the occurrence of \( F \) does not affect the likelihood of \( E \),
  and vice versa.

- **Ex**
  \[
  P\left(10^{th \text{ flip}}=H \mid \text{ first 9 flips } = H \right) = P(E) = \frac{1}{2} \quad (E \perp F)
  \]

- **Rewrite (\times):**
  \[
  \frac{P(E \cap F)}{P(F)} = P(E), \quad \frac{P(F \cap E)}{P(E)} = P(F) \quad \iff \quad P(E \cap F) = P(E) \cdot P(F).
  \]

**Def** Events \( E, F \) are independent \( (E \perp F) \) if
\[
P(E \cap F) = P(E) \cdot P(F).
\]

- In Example above, \( P\{\text{both flips } = H\} = \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} \Rightarrow \text{indep.} \)

**Warning:** independent \( \neq \) mutually exclusive!

- In fact, mutually exclusive sets are typically dependent

**Example:**
\[
R_1 = \text{"finding a missing hiker in Region 1"}
R_2 = \text{"Region 2"}
\]
are mutually exclusive, very dependent: \( P(R_1 | R_2) = 0 \)
Independence of 3 events:

Def. E, F, G are (jointly) independent if

\[
P(E \cap F) = P(E)P(F); \quad P(E \cap G) = P(E)P(G); \quad P(F \cap G) = P(F)P(G).
\]

\[P(E \cap F \cap G) = P(E)P(F)P(G).\]

"pairwise independent"

Ex: Pairwise independence $\not\Rightarrow$ independence:

Experiment: pick a number from \{1, 2, 3, 4\} at random and with uniform probability. Events:

- \( E = \text{"1 or 2"}, \quad F = \text{"1 or 3"}, \quad G = \text{"2 or 3"} \)

- \( P(E) = P(F) = P(G) = \frac{2}{4} = \frac{1}{2} \)

- \( P(E \cap F) = \frac{1}{4} = P(E)P(F) \Rightarrow E \perp F \)

Similarly for the other pairs $\Rightarrow$ pairwise indep

- But \( P(E \cap F \cap G) = 0 \neq P(E)P(F)P(G) \Rightarrow \text{not indep} \)

For \( k \) number of events:

Def. \( E_1, E_2, \ldots \) are (jointly) independent if

\[
P\left( \bigcap_{i \in I} E_i \right) = \prod_{i \in I} P(E_i) \quad \forall \text{subset } I.
\]
Prop (Independence is stable)

If \( E_1, E_2, \ldots, F_1, F_2, \ldots \) are all independent, then

\( \forall \) event formed of \( E_1, E_2, \ldots \) is independent of

\( \forall \) event formed of \( F_1, F_2, \ldots \)

Using \( \cap, \cup, \text{complementary} \)

First, check a couple of special cases:

If \( E, F, G \) are indep then:

(a) Complements: \( E \perp F^c \)

\[
P(E) = P(EnF) + P(EnF^c) \Rightarrow P(EnF^c) = P(E) - P(E)P(F)
\]

\[
= P(E)(1 - P(F)) = P(E)P(F^c)
\]

\( P(E) \cdot P(F) \) by independence

(b) Intersections: \( E \perp (F \cap G) \)

\[
P(En(F \cap G)) = P(EnF \cap G) = P(E)P(F)P(G) \text{ (indep)}
\]

\[
= P(E)P(F \cap G)
\]

\( E \perp G \)

(c) Any other set operations can be built from complements & intersec:

union \( F \cup G = (F^c \cap G^c)^c \text{ (de Morgan)} \)

iterate.