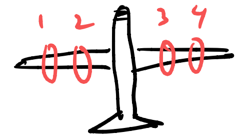
S3:E7

Ex A Boeing 747 can fly if at least one engine on each wing works.



During a flight an engine can fail with probability p. The engines fail independently.

Find the prob. that a plane can fly.

E_i = "engine i works". Independent. $P(E_i) = 1-p$

$$F = \text{"plane flies"} = \underbrace{(E_1 \cup E_2)}_{\text{at least 1 engine on left wing works}} \cap \underbrace{(E_3 \cup E_4)}_{\text{" — " right wing}}$$

independent by stability of independence

$$P(F) = P(E_1 \cup E_2) \cdot P(E_2 \cup E_3)$$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - \underbrace{P(E_1 \cap E_2)}_{= P(E_1)\,P(E_2)} \qquad \text{(IEP)} \quad \text{(independence)}$$

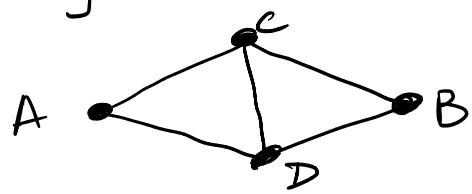
$$= (1-p) + (1-p) - (1-p)^2 = 1 - p^2$$

Alternatively,
$P(E_1 \cup E_2) = 1 - P(E_1^c \cap E_2^c)$
$= 1 - P(E_1^c)\,P(E_2^c) = 1 - p^2$

Similarly, $P(E_3 \cup E_4) =$ ⤴

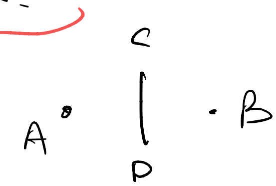
$$\Rightarrow P(F) = (1-p^2)^2$$

Ex 4 mountain villages are connected by 5 roads:



On a snowy day, each road is open with probability p, independently of the others.

Find the prob. that $\underbrace{\text{one can get from A to B}}_{E}$



Condition on the state of the CD road

LTP $\Rightarrow$

$$P(E) = P(E|\overset{\text{CD road is open}}{CD})\underbrace{P(CD)}_{=p} + P(E|\overset{\text{closed}}{CD^c})\underbrace{P(CD^c)}_{=1-p} \qquad (*)$$

① $P(E|CD) = ?$

If road CD is open, $E \Leftrightarrow$ Both A and B are connected to CD

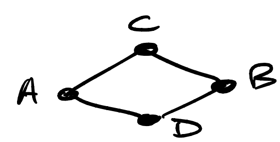
A is connected to CD $\Leftrightarrow$ $\underset{\text{prob. } p}{\text{AC open}}$ or $\underset{p}{\text{AD open}}$, or $\underset{p^2}{\text{Both}}$

I.E.P : prob $= p + p - p^2 = 2p - p^2$

Same for B. Independence $\Rightarrow$ $P(E|CD) = (2p-p^2)^2$.

② $P(E|CD^c) = ?$

If CD road is closed, network looks like this:



$E \Leftrightarrow$ $\underbrace{\text{route AC}-\text{CB open}}_{\text{prob} = p^2}$, or $\underbrace{\text{route AD}-\text{DB open}}_{p^2}$, or $\underbrace{\text{Both}}_{p^4}$

I.E.P: $P(E|CD^c) = p^2 + p^2 - p^4 = 2p^2 - p^4$.

Combine $\overset{(*)}{\Longrightarrow}$ $P(E) = \boxed{(2p-p^2)^2 p + (2p^2 - p^4)(1-p)}$

Remark: Random graphs (Erdős-Rényi), connectivity