S3: E8

Ex Two players take turns flipping a coin. The first player to obtain a head wins. What is the prob. that $\underbrace{\text{the player who starts wins}}_{E}$?

Condition on the first flip

player 1 wins   player 1 flips H   or T

$$P(E) = \underbrace{P(E|H)}_{1}\,\underbrace{P(H)}_{\frac{1}{2}} + \underbrace{P(E|T)}_{\text{game resets, player 2 starts}}\,\underbrace{P(T)}_{\frac{1}{2}}$$

$P(E|T) = P(\text{the player who starts loses})$

$= 1 - P(E)$

$$\Rightarrow P(E) = \frac{1}{2} + (1 - P(E)) \cdot \frac{1}{2}$$

Solving gives $P(E) = 2/3$

Ex (The problem of points)

Teams A and B play against each other continually.
The first team that wins 5 games wins the tournament
($\Rightarrow$ ≤ 9 games total).
Team A has prob. 0.6 to win $\forall$ given game, independently.
What is the prob. that team A wins the tournament?

A wins the tournament if

$$E_{5,5} = \{A \text{ wins } 5 \text{ games before } B \text{ wins } 5 \text{ games}\}$$

Condition on the outcome of 1^{st} game:

$$P(E_{5,5}) = \underbrace{P(E_{5,5} \mid A_1)}_{= P(E_{4,5}),\ \text{prob. that A wins 4 games before B wins 5}} \underbrace{P(A_1)}_{0.6} + \underbrace{P(E_{5,5} \mid B_1)}_{= P(E_{5,4}),\ \text{prob. that A wins 5 games before B wins 4}} \underbrace{P(B_1)}_{= 0.4}$$

$$\Rightarrow P(E_{5,5}) = 0.6\, P(E_{4,5}) + 0.4\, P(E_{5,4}).$$

More generally, denoting $P_{n,m} = P(E_{n,m})$, we get

$$\begin{cases} P_{n,m} = 0.6\, P_{n-1,m} + 0.4\, P_{n,m-1} \quad \forall\ 1 \leq n,m \leq 5 \\ P_{n,0} = 0; \quad P_{0,m} = 1 \end{cases}$$

- System of linear equations. Can be solved computationally (Matlab), or by induction, or even analytically (Pascal Δ)
- In particular, $P_{5,5} = 0.73$ Ans: 73%