S4 : E2

- **Recall** (S4 : E1) : a r.v. X is an assignment : outcome $\mapsto$ number

  i.e. X is a function $S \to \mathbb{R}$.

Examples: (a) $X = \#$ heads in 3 coin flips

$$S = \{TTT,\ TTH,\ THT,\ THH,\ HTT,\ HTH,\ HHT,\ HHH\}$$

$$X: \quad 0 \quad 1 \quad 1 \quad 2 \quad 1 \quad 2 \quad 2 \quad 3$$

(b) $X =$ age of a randomly chosen person

$S = \{\text{people}\}$; $\quad X$: person $\mapsto$ his/her age

(c) $X = \#$ children in a randomly chosen family

$S = \{\text{families}\}$ $\quad X$: family $\mapsto$ # children

- **Distribution** of X = knowledge of what values X takes, with what probabilities

If X is a discrete r.v (# values finite or countable), distribution is captured by:

probability mass function (pmf) $\quad p(x) = P\{X = x\}$
$= P\{s \in S : X(s) = x\}$

Ex (a): $p(0) = \frac{1}{8}$, $p(1) = \frac{3}{8}$, $p(2) = \frac{3}{8}$, $p(3) = \frac{1}{8}$

Today: Expected value, a.k.a. "expectation", "mean", "average"

— a single # that best describes X

- **Ex** Study of 836 families.

502	families :	no children
140	:	1 child
127	:	2
67	:	3

What is the average number of children in a family?

$$\text{Average} = \frac{\text{total \#children}}{\text{total \#families}} = \frac{0\cdot 502 + 1\cdot 140 + 2\cdot 127 + 3\cdot 67}{836} \approx 0.69$$

- Probabilistic view:

experiment = choose a family at random (uniform distribution)

r.v. X = # children in this family

pmf: $p(0) = \frac{502}{836}$, $p(1) = \frac{140}{836}$, $p(2) = \frac{127}{826}$, $p(3) = \frac{67}{836}$

$\Rightarrow$ Average $= 0\cdot p(0) + 1\cdot p(1) + 2\cdot p(2) + 3\cdot p(3) =: E[X]$.

- **Def** The expected value of a discrete r.v. X is

$$\boxed{E[X] = \sum_x x\, P\{X=x\} = \sum_i x_i\, p(x_i)}$$

Intuition :

- if all N values are equally likely $\Rightarrow p(x_i) = \frac{1}{N}$

$\Rightarrow E[X] = \frac{1}{N}\sum_{i=1}^{N} x_i$, arithmetic mean

- In general, $\sum_{i=1}^{N} x_i \, p(x_i)$ is a weighted average

more weight $\rightarrow$ more likely values.

- $\simeq$ center of mass (particles at x_i, weights $p(x_i)$)

Ex X = # heads in 3 coin flips

$p(0) = \frac{1}{8}$, $p(1) = \frac{3}{8}$, $p(2) = \frac{3}{8}$, $p(3) = \frac{1}{8}$

$E[X] = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 1.5$

Ex (Lottery) 6 out of 49 (in any order)

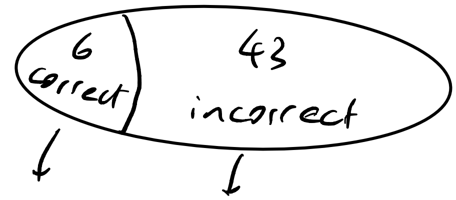
All 6 correct $\Rightarrow$ prize $= \$1,200,000$

5 correct $\Rightarrow$ prize $= \$800$

4 correct $\Rightarrow$ prize $= \$35$

≤ 3 correct $\Rightarrow$ no prize.

Expected winnings $= ?$



proof: $p(1,200,000) = P\{\text{all 6 correct}\} = \dfrac{1}{\binom{49}{6}}$

$$p(800) = P\{5 \text{ correct}, 1 \text{ not}\} = \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}}$$

$$p(35) = P\{4 \text{ correct}, 2 \text{ not}\} = \frac{\binom{6}{4}\binom{43}{2}}{\binom{49}{6}}$$

$$p(0) = 1 - p(1,200,000) - p(800) - p(35).$$

$$E[X] = 1,200,000 \cdot p(1,200,000) + 800 \cdot p(800) + 35 \cdot p(35) + 0 \cdot p(0)$$

$$\approx 0.13.$$

Ans: 13¢ $\leftarrow$ would be a fair price for a lottery ticket.