$\boxed{S4 : E3}$

PROPERTIES OF EXPECTATION

Prop $\mathbb{E}(a) = a$ if $a \in \mathbb{R}$ is a constant.

Proof: $X = a$ takes value a with prob. 1 $\Rightarrow E[X] = 1 \cdot a = a$. QED

$$\boxed{\text{THM (linearity)} \quad \mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y] \quad \forall \text{ r.v.'s } X, Y \text{ and } \forall \text{ constants } a, b \in \mathbb{R}}$$

Proof for discrete r.v.'s.

If X takes values x_i with prob. p_i

and Y takes values y_j with prob. q_j,

then $aX + bY$ takes values $ax_i + by_j$ with prob. $P_{ij} = P\{X = x_i, Y = y_j\}$

possibly repeated (ok in the def.)

$$\mathbb{E}[X+Y] = \sum_{i,j} (ax_i + by_j)\, P_{ij} = \sum_i a x_i \underbrace{\Big(\sum_j P_{ij}\Big)}_{= p_i} + \sum_j b y_j \underbrace{\Big(\sum_i P_{ij}\Big)}_{= q_j}$$

$$= a \sum_i x_i p_i + b \sum_j y_j q_j$$

$$= a\,\mathbb{E}[X] + b\,\mathbb{E}[Y].$$

QED.

By induction $\Rightarrow$

$$\boxed{\text{Cor} \quad \mathbb{E}\Big[\sum_i X_i\Big] = \sum_i \mathbb{E}[X_i]}$$

Ex $X = \#$ heads in 3 coin flips $= X_1 + X_2 + X_3$ ← #heads in flip $1 \in \{0,1\}$

$$E[X] = E[X_1] + E[X_2] + E[X_3].$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1.5$$

$E[X_i] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

Ex (St. Petersburg Paradox)

The initial stake (money in the pot) is \$2.

\$2

The player tosses a fair coin.

If T ⇒ game ends, the player wins whatever is in the pot.

If H ⇒ stake doubles, game continues.

What would be a fair price to pay the casino for entering this game?

X = player's winnings.

$p(2) = 1/2, \quad p(4) = 1/4, \ldots$

$$E[X] = 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \ldots = \infty$$

Tosses	Winnings \$	Prob.	
T	2	1/2	1/2
HT	4	1/4	1/2
HHT	8	1/8	
HHHT	16	1/16	
...	...	...	

Ans: $\forall$ price.

Remarks:
- Volatility, financial bubbles.
- Median is more stable. $Med(X) = M$ if
$$P\{X \le M\} \ge \frac{1}{2}, \quad P\{X \ge M\} \ge \frac{1}{2}.$$
- In the St Peter's paradox, $Med(X) \in [2, 4]$